

Spectral and scattering theory of fourth order differential operators

D. R. Yafaev

To Mikhail Shlëmovich Birman on his 80-th birthday

ABSTRACT. An ordinary differential operator of the fourth order with coefficients converging at infinity sufficiently rapidly to constant limits is considered. Scattering theory for this operator is developed in terms of special solutions of the corresponding differential equation. In contrast to equations of second order “scattering” solutions contain exponentially decaying terms. A relation between the scattering matrix and a matrix of coefficients at exponentially decaying modes is found. In the second part of the paper the operator D^4 on the half-axis with different boundary conditions at the point zero is studied. Explicit formulas for basic objects of the scattering theory are found. In particular, a classification of different types of zero-energy resonances is given.

1. Introduction

1.1. General scattering theory for differential operators does not depend on the order of operators (see [4] for the trace class approach and [10] for the smooth approach). Suppose that coefficients of a differential operator H converge sufficiently rapidly to constant values at infinity, and let H_0 be the operator with these constant coefficients. Then the wave operators $W_{\pm}(H, H_0)$ for the pair H_0, H exist and are complete, and the corresponding scattering matrix is a unitary operator. The operator H does not have the singular continuous spectrum, and its point spectrum might accumulate only at thresholds (critical values of the symbol of the operator H_0). Moreover, an expansion theorem in eigenfunctions of the operator H is true.

However the behavior of eigenfunctions of the continuous spectrum at infinity is essentially simpler for differential operators of the second order than for higher order differential operators. This is intimately related to the fact that higher order differential operators might have eigenvalues embedded in the continuous spectrum.

2000 *Mathematics Subject Classification.* 34B25, 35P25, 47A40.

Key words and phrases. one-dimensional differential operators, fourth order, exponentially decaying modes, scattering matrix, spectral shift function, zero-energy resonances.

1.2. In this paper we consider one-dimensional differential operators. For definiteness, we choose operators of the fourth order. Set $H_0 = D^4$ and

$$(1.1) \quad H = D^4 + Dv_1(x)D + v_0(x), \quad D = -id/dx,$$

where the functions v_0 and v_1 are real and satisfy a short-range assumption

$$(1.2) \quad v_j \in L_1(\mathbb{R}), \quad j = 0, 1.$$

A smoothness of the function $v_1(x)$ is not required. We are interested in construction of the wave operators and of the scattering matrix in terms of solutions of the corresponding differential equation

$$(1.3) \quad (u'''(x) - v_1(x)u'(x))' + v_0(x)u(x) = \lambda u(x), \quad \lambda > 0.$$

This construction turns out to be more complicated than for equations of the second order. Indeed, the “free” equation $u^{(4)}(x) = \lambda u(x)$ has solutions e^{ikx} , e^{-ikx} , e^{-kx} and e^{kx} where $\lambda = k^4$, $k > 0$. If, for example, the coefficients $v_0(x)$ and $v_1(x)$ are compactly supported, then every solution of equation (1.3) is a linear combination of these exponentials for large positive and large negative x . Eigenfunctions $\psi_j(x, \lambda)$, $j = 1, 2$, of the operator H (its continuous spectrum has multiplicity two) are special solutions of equation (1.3). It is natural to expect that they do not contain exponentially increasing terms and that the scattering matrix is determined only by coefficients at oscillating terms e^{ikx} and e^{-ikx} . We justify this conjecture under general short-range assumption (1.2).

Furthermore, if the coefficients $v_1(x)$ and $v_0(x)$ decay super-exponentially at infinity, then it is possible to distinguish exponentially decreasing terms in the asymptotics of the functions $\psi_j(x, \lambda)$, $j = 1, 2$, as $x \rightarrow \pm\infty$. Our main observation in the first part of the paper is that the coefficients at these terms determine the scattering matrix. This might be eventually of interest for a study of the (inverse) problem of a reconstruction of the coefficients $v_0(x)$ and $v_1(x)$ from scattering data. We refer to [2] for a comprehensive study of the inverse problem.

Recall that for differential operators of the second order the eigenfunctions of the continuous spectrum can be constructed (see [14]) with a help of Volterra integral equations. This procedure seems not to work for operators of higher order. Therefore we use a general scheme of scattering theory (see, e.g., [11, 16]) relying on the Lippmann-Schwinger equation.

In Section 2 we present a stationary approach to scattering theory for the operator H . This approach is quite general and, up to some technical details, works for multi-dimensional differential operators of an arbitrary order (see [10]). The absence of the singular continuous spectrum is verified in Section 3. Here instead of Agmon’s bootstrap arguments [1], a method specific for ordinary differential equations is used. After this prerequisite, we study in Section 4 asymptotic behavior of eigenfunctions of the continuous spectrum. The relation between the scattering matrix and the coefficients at exponentially decreasing modes is formulated in Theorem 4.8.

1.3. The second part of the paper (Sections 5, 6 and 7) is devoted to a study of the operator $H = D^4$ in the space $L_2(\mathbb{R}_+)$ with some self-adjoint boundary conditions at the point $x = 0$. This operator can be compared with a well known Hamiltonian D^2 with a boundary condition

$$(1.4) \quad u'(0) = \alpha u(0), \quad \alpha = \bar{\alpha}.$$

This boundary condition is interpreted as a point interaction at the point $x = 0$. The point interaction is a good approximation to a perturbation by a potential for low (but not for high) energies.

We shall write down explicit (although not very simple) formulas for basic objects of scattering theory for the operator H , such as the resolvent kernel, eigenfunctions, the scattering matrix, the perturbation determinant, the spectral shift function and so forth. This model seems to be of interest because it allows us to analyse a behavior of different objects at zero energy as well as at a positive eigenvalue of the operator H . In particular, we discuss different types of zero-energy resonances.

There exist several definitions of zero-energy resonances which are essentially (but not completely) equivalent. According to a general variational definition (see [3]) an operator H has a zero-energy resonance if for a small negative perturbation an additional negative eigenvalue appears. Of course this definition depends not only on H but also on a class of perturbations. Other definitions (see [15] or [18]) are adapted to differential operators. Thus, the Schrödinger operator $H = -\Delta + v(x)$ has a zero-energy resonance if the kernel of its resolvent $R(z) = (H - z)^{-1}$ has a singularity $\varphi(x)\overline{\varphi(x')}(-z)^{-1/2}$ as $|z| \rightarrow 0$. Here $\varphi(x)$ is a solution of the equation $H\varphi = 0$ which is bounded at infinity (in the one-dimensional case). The existence of such solutions gives still another criterium for the appearance of zero-energy resonances. Zero-energy resonances can be considered as a weakened version of bound states (for zero energy) and are often called half-bound states. This point of view is confirmed by the behavior of the spectral shift function $\xi(\lambda)$ at the point $\lambda = 0$. Consider, for example, the operators H_0 and $H(\alpha)$ corresponding to the differential expression D^2 in the space $L_2(\mathbb{R}_+)$ with boundary conditions $u(0) = 0$ and (1.4), respectively. The operator $H(\alpha)$ has a zero-energy resonance if and only if $\alpha = 0$. The spectral shift function $\xi(\lambda)$ for the pair $H_0, H(\alpha)$ is continuous at $\lambda = 0$ if $\alpha \neq 0$ and it has the jump $-1/2$ if $\alpha = 0$. This should be compared with the fact that the jump of $\xi(\lambda)$ equals -1 at an eigenvalue of H .

In Section 7 we analyse in some details zero-energy resonances for the operator $H = D^4$ with different boundary conditions at the point $x = 0$. It turns out that, compared to second order, for fourth order differential operators their notion acquires some new features although the variational definition remains of course valid. A more detailed classification is given in terms of singularities of the resolvent at the point $z = 0$, of a behavior of solutions (which are of course all polynomials of degree 3) of the equation $u^{(4)}(x) = 0$ satisfying the boundary condition and of a jump of the corresponding spectral shift function at the point $\lambda = 0$. This analysis shows that it is natural to introduce now 1/4- and 3/4-bound states. Here we mention only that the operator H has a 1/4-bound state (a 3/4-bound state) if a non-trivial linear function (a constant) satisfies the boundary conditions at $x = 0$.

Thanks are due to V. Suhanov for a useful discussion.

2. General scattering theory

2.1. Under the assumption

$$\sup_{x \in \mathbb{R}} \int_x^{x+1} (|v_1(y)| + |v_0(y)|) dy < \infty$$

we have that, for all $\varepsilon > 0$ and all functions u from the Sobolev class $H^2(\mathbb{R})$,

$$(2.1) \quad \begin{aligned} & \int_{-\infty}^{\infty} (|v_1(x)||u'(x)|^2 + |v_0(x)||u(x)|^2) dx \\ & \leq \varepsilon \int_{-\infty}^{\infty} |u''(x)|^2 dx + C(\varepsilon) \int_{-\infty}^{\infty} |u(x)|^2 dx. \end{aligned}$$

Here and below C denotes different positive constants whose precise values are of no importance. Therefore the quadratic form

$$h[u, u] = \int_{-\infty}^{\infty} (|u''(x)|^2 + v_1(x)|u'(x)|^2 + v_0(x)|u(x)|^2) dx$$

is semibounded from below and is closed on $H^2(\mathbb{R})$. Thus, it defines a self-adjoint operator H in the Hilbert space $\mathcal{H} = L_2(\mathbb{R})$ with domain $\mathcal{D}(H) \subset H^2(\mathbb{R})$. This operator corresponds to formal differential expression (1.1).

Note that the operators $H_0 = D^4 : H^2(\mathbb{R}) \rightarrow H^{-2}(\mathbb{R})$ and, by virtue of estimate (2.1),

$$(2.2) \quad V = Dv_1(x)D + v_0(x) : H^2(\mathbb{R}) \rightarrow H^{-2}(\mathbb{R})$$

are bounded operators. It can easily be shown that a function $u \in H^2(\mathbb{R})$ belongs to $\mathcal{D}(H)$ if and only if $H_0u + Vu \in L_2(\mathbb{R})$; in this case $Hu = H_0u + Vu$. It follows that, for $u \in \mathcal{D}(H)$, the function $u'''(x) - v_1(x)u'(x)$ is absolutely continuous and

$$(Hu)(x) = (u'''(x) - v_1(x)u'(x))' + v_0(x)u(x).$$

Let us discuss main steps of a construction of scattering theory for the operator H . The resolvent $R_0(z) = (H_0 - z)^{-1}$, $z \in \mathbb{C} \setminus [0, \infty)$, of the operator H_0 can be calculated explicitly.

LEMMA 2.1. *Let $z \in \mathbb{C} \setminus [0, \infty)$ and $\zeta^4 = z$, $\arg \zeta \in (0, \pi/2)$. Then*

$$(2.3) \quad (R_0(z)f)(x) = \frac{1}{4\zeta^3} \int_{-\infty}^{\infty} (ie^{i\zeta|x-y|} - e^{-\zeta|x-y|})f(y)dy.$$

PROOF. Using the Fourier transform, we see that

$$(R_0(z)f)(x) = 2\pi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(x-y)}(k^4 - z)^{-1}f(y)dkdy.$$

Therefore the resolvent kernel equals

$$R_0(x, y; z) = 2\pi^{-1} \int_{-\infty}^{\infty} e^{ik|x-y|}(k^4 - z)^{-1}dk.$$

This integral can be complemented in the upper half-plane by a big half-circle and then calculated by residues at the points ζ and $i\zeta$. \square

Let us denote by Π the complex plane cut along $[0, \infty)$ (including upper and lower edges). According to Lemma 2.1 the resolvent kernel $R_0(x, y; z)$ is a continuous function of $z \in \Pi$ with an exception of the point $z = 0$. The following two results are immediate consequences of explicit formula (2.3).

PROPOSITION 2.2. *If $f \in L_1(\mathbb{R})$, then for all $z \in \Pi$, $z \neq 0$, the function $R_0(z)f \in C^3(\mathbb{R})$ and $(R_0(z)f)'''(x)$ is an absolutely continuous function.*

PROPOSITION 2.3. *Let G_j be the operator of multiplication by a function $g_j \in L_2(\mathbb{R})$, $j = 0, 1$. Then the operator-valued functions*

$$G_0 D^l R_0(z) G_1, \quad l = 0, 1, 2, 3,$$

depend in the Hilbert-Schmidt norm continuously on $z \in \Pi$, $z \neq 0$.

To extend the latter result to the resolvent $R(z) = (H - z)^{-1}$ of the operator H , we proceed from the resolvent identity

$$(2.4) \quad R(z) = R_0(z) - R_0(z) V R(z) = R_0(z) - R(z) V R_0(z), \quad \text{Im } z \neq 0,$$

where V is operator (2.2). Let G_j and Ω_j be operators of multiplication by the functions $\sqrt{|v_j(x)|}$ and $\text{sgn } v_j(x)$, respectively. We introduce an auxiliary space $\mathcal{G} = \mathcal{H} \oplus \mathcal{H}$ and define (bounded) operators $\mathbf{G}_0, \mathbf{G} : H^2(\mathbb{R}) \rightarrow \mathcal{G}$ by formulas¹

$$(2.5) \quad \mathbf{G}_0 = (\Omega_0 G_0, \Omega_1 G_1 D), \quad \mathbf{G} = (G_0, G_1 D).$$

It follows from equality (2.2) that $V = \mathbf{G}^* \mathbf{G}_0 = \mathbf{G}_0^* \mathbf{G}$. The resolvent identity (2.4) implies that

$$(I + \mathbf{G}_0 R_0(z) \mathbf{G}^*)(I - \mathbf{G}_0 R(z) \mathbf{G}^*) = (I - \mathbf{G}_0 R(z) \mathbf{G}^*)(I + \mathbf{G}_0 R_0(z) \mathbf{G}^*) = I.$$

Hence the inverse operator $(I + \mathbf{G}_0 R_0(z) \mathbf{G}^*)^{-1}$ exists and is bounded so that using again (2.4) we obtain the representation

$$(2.6) \quad R(z) = R_0(z) - R_0(z) \mathbf{G}^* (I + \mathbf{G}_0 R_0(z) \mathbf{G}^*)^{-1} \mathbf{G}_0 R_0(z).$$

Thus, the resolvent $R(z)$ for $\text{Im } z \neq 0$ considered as a mapping from $H^{-2}(\mathbb{R})$ to $H^2(\mathbb{R})$ is a bounded operator.

It follows from Proposition 2.3 that under assumption (1.2) the operator-valued function $\mathbf{G}_0 R_0(z) \mathbf{G}^*$, analytic for $\text{Im } z \in \mathbb{C} \setminus [0, \infty)$, is continuous in the Hilbert-Schmidt norm for $z \in \Pi$ except the point $z = 0$. Therefore according to the analytic Fredholm alternative (see, e.g., [16]) the set $\mathcal{N} \in \mathbb{R}_+$ where at least one of two homogeneous equations

$$(2.7) \quad \mathbf{f} + \mathbf{G}_0 R_0(\lambda \pm i0) \mathbf{G}^* \mathbf{f} = 0, \quad \mathbf{f} = (f_0, f_1),$$

has a non-trivial solution $\mathbf{f} \in \mathcal{G}$ is closed and has the Lebesgue measure zero. The operator-valued function $(I + \mathbf{G}_0 R_0(z) \mathbf{G}^*)^{-1}$ of $z \in \Pi$ is continuous in norm except points from the set $\mathcal{N} \cup \{0\}$. Therefore equation (2.6) leads to the following result.

THEOREM 2.4. *Let assumption (1.2) hold, and let G_j be the same operators as in Proposition 2.3. Then the operator-valued functions $G_1 R(z) G_2$, $G_1 D R(z) G_2$ and $G_1 D R(z) D G_2$ of $z \in \Pi$ are continuous in the Hilbert-Schmidt norm except points from the set $\mathcal{N} \cup \{0\}$. The set $\Lambda = \mathbb{R}_+ \setminus \mathcal{N}$ is open and has full Lebesgue measure.*

COROLLARY 2.5. *The spectrum of the operator H on the set Λ is absolutely continuous.*

We denote by $P^{(c)}$ the orthogonal projector on the absolutely continuous subspace $\mathcal{H}^{(c)}$ of the operator H .

2.2. Given Theorem 2.4, an expansion in eigenfunctions of the operator H , a formula representation of the scattering matrix, etc., are consequences of general results of scattering theory (see [6] or [16]).

¹Although it is more convenient to write vectors as rows, we regard them as columns as far as matrix multiplication is concerned.

Denote by $E_0(\lambda)$ the spectral family of the operator H_0 . In the momentum representation the operator $E_0(\lambda)$ acts as multiplication by the characteristic function $\chi_\lambda(\xi)$ of the interval $(-\sqrt[4]{\lambda}, \sqrt[4]{\lambda})$, that is

$$(\widehat{E_0(\lambda)f_0})(\xi) = \chi_\lambda(\xi)\hat{f}_0(\xi)$$

where $\hat{f}_0(\xi)$ is the Fourier transform of the function $f_0(x)$. It follows that for $f_0 \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ the function $(E_0(\lambda)f_0, f_0)$ belongs to the class $C^1(\mathbb{R}_+)$ and

$$(2.8) \quad d(E_0(\lambda)f_0, f_0)/d\lambda = 4^{-1}\lambda^{-3/4}(|\hat{f}_0(\sqrt[4]{\lambda})|^2 + |\hat{f}_0(-\sqrt[4]{\lambda})|^2).$$

Next we construct the canonical spectral representation of the operator H_0 . We set

$$(2.9) \quad \Gamma_0(\lambda)f_0 = 2^{-1}\lambda^{-3/8}(\hat{f}_0(\lambda^{1/4}), \hat{f}_0(-\lambda^{1/4})), \quad \Gamma_0(\lambda) : L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \rightarrow \mathbb{C}^2,$$

and define the operator $\mathcal{F}_0 : L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}_+; \mathbb{C}^2)$ by the formula $(\mathcal{F}_0 f)(\lambda) = \Gamma_0(\lambda)f$. This operator extends by continuity to a unitary mapping $\mathcal{F}_0 : \mathcal{H} \rightarrow L_2(\mathbb{R}_+; \mathbb{C}^2)$. Then $\mathcal{F}_0 H_0 = A\mathcal{F}_0$ where A is the operator of multiplication by λ in the space $L_2(\mathbb{R}_+; \mathbb{C}^2)$.

Now we discuss generalizations of these objects for the operator H . Let us set

$$(2.10) \quad \Gamma_\pm(\lambda)f = \Gamma_0(\lambda)(I - VR(\lambda \pm i0))f, \quad f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}), \quad \lambda \in \Lambda.$$

According to Theorem 2.4, $\Gamma_\pm(\lambda)f$ is a continuous function of $\lambda \in \Lambda$. A proof of the following result can be found in [6] or [16].

THEOREM 2.6. *Let assumption (1.2) hold. Define the mapping \mathcal{F}_\pm on the set $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ by equalities (2.10) and*

$$(2.11) \quad (\mathcal{F}_\pm f)(\lambda) = \Gamma_\pm(\lambda)f.$$

This mapping extends by continuity to a bounded operator $\mathcal{F}_\pm : \mathcal{H} \rightarrow L_2(\mathbb{R}_+; \mathbb{C}^2)$, satisfies the relations

$$(2.12) \quad \mathcal{F}_\pm \mathcal{F}_\pm^* = I, \quad \mathcal{F}_\pm^* \mathcal{F}_\pm = P^{(c)}$$

and diagonalizes H , that is

$$(2.13) \quad \mathcal{F}_\pm H = A\mathcal{F}_\pm.$$

Time-dependent wave operators for the pair H_0, H are defined as strong limits

$$(2.14) \quad W_\pm = W_\pm(H, H_0) = \text{s-lim}_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0 t}.$$

Recall that, by the spectral theorem,

$$R(\lambda \pm i\varepsilon) = \pm \int_0^\infty e^{-\varepsilon t \pm i\lambda t} e^{\mp iHt} dt$$

so that, by the Parseval identity,

$$(2.15) \quad 2\varepsilon \int_0^\infty e^{-2\varepsilon t} (e^{\mp iHt} f_0, e^{\mp iHt} f) dt = \pi^{-1} \varepsilon \int_{-\infty}^\infty (R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)f) d\lambda$$

for all $f_0, f \in \mathcal{H}$.

Let us show that this expression has a limit as $\varepsilon \rightarrow 0$. This entails the existence of the weak wave operators understood, moreover, in the Abelian sense. Such wave operators are defined by the limit of the left-hand side of (2.15). It suffices to verify the existence of the limit for $f_0, f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$.

Let us consider the right-hand side of (2.15). Note that, by the resolvent identity (2.4),

$$\pi^{-1}\varepsilon(R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)f) = (\delta_\varepsilon(H_0 - \lambda)f_0, (I - VR(\lambda \pm i\varepsilon))f),$$

where

$$\delta_\varepsilon(H_0 - \lambda) = (2\pi i)^{-1}(R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon))$$

is an “approximate” operator-valued delta-function. It follows from the spectral theorem, standard properties of the Cauchy type singular integrals and formulas (2.8), (2.9) that

$$(2.16) \quad \lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon(H_0 - \lambda)f_0, f) = d(E_0(\lambda)f_0, f)/d\lambda = \langle \Gamma_0(\lambda)f_0, \Gamma_0(\lambda)f \rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{C}^2 . Therefore according to Proposition 2.3 we have

$$(2.17) \quad \lim_{\varepsilon \rightarrow 0} \mathbf{G}_0 \delta_\varepsilon(H_0 - \lambda)f_0 = \mathbf{G}_0 \Gamma_0^*(\lambda) \Gamma_0(\lambda)f_0.$$

The convergence in (2.16) and (2.17) is uniform on compact intervals of \mathbb{R}_+ . Furthermore, Theorem 2.4 ensures that there exists

$$(2.18) \quad \lim_{\varepsilon \rightarrow 0} \mathbf{G}R(\lambda \pm i\varepsilon)f = \mathbf{G}R(\lambda \pm i0)f.$$

Combining relations (2.16), (2.17) and (2.18), we obtain that

$$(2.19) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon(H_0 - \lambda)f_0, (I - VR(\lambda \pm i\varepsilon))f) &= \langle \Gamma_0(\lambda)f_0, \Gamma_0(\lambda)f \rangle \\ -(\mathbf{G}_0 \Gamma_0^*(\lambda) \Gamma_0(\lambda)f_0, \mathbf{G}R(\lambda \pm i0)f) &= \langle \Gamma_0(\lambda)f_0, \Gamma_\pm(\lambda)f \rangle. \end{aligned}$$

The convergence in (2.18) and (2.19) is uniform with respect to λ from compact intervals of the set Λ .

It remains to justify a passage to the limit $\varepsilon \rightarrow 0$ in the integral in the right-hand side of (2.15). Note that, by the Schwarz inequality, for any Borel set $X \subset \mathbb{R}$

$$(2.20) \quad \begin{aligned} &\left| \int_X \varepsilon(R_0(\lambda \pm i\varepsilon)f_0, R(\lambda \pm i\varepsilon)f) d\lambda \right|^2 \\ &\leq \int_X \varepsilon \|R_0(\lambda \pm i\varepsilon)f_0\|^2 d\lambda \int_{-\infty}^{\infty} \varepsilon \|R(\lambda \pm i\varepsilon)f\|^2 d\lambda \\ &= \pi^2 \int_X (\delta_\varepsilon(H_0 - \lambda)f_0, f_0) d\lambda \|f\|^2. \end{aligned}$$

Since the function $(\delta_\varepsilon(H_0 - \lambda)f_0, f)$ is the Poisson integral of function (2.16) belonging to the space $L_1(\mathbb{R})$, we see (see, e.g., [9]) that the convergence in (2.16) holds true in the sense of $L_1(\mathbb{R})$. Therefore the right-hand side of (2.20) tends to zero as $|X| \rightarrow 0$ or $X = (N, \infty)$ and $N \rightarrow \infty$ uniformly with respect to $\varepsilon \in (0, 1)$. Moreover, it tends to zero as $\varepsilon \rightarrow 0$ if $X = (-\infty, 0)$.

Thus, we have shown that the integral in the right-hand side of (2.15) has the limit which equals the integral of function (2.19) over $\lambda \in \mathbb{R}_+$. It follows that there exists the limit

$$(2.21) \quad \lim_{\varepsilon \rightarrow 0} 2\varepsilon \int_0^\infty e^{-2\varepsilon t} (e^{\mp itH_0} f_0, e^{\mp itH} f) dt = \int_0^\infty \langle \Gamma_0(\lambda)f_0, \Gamma_\pm(\lambda)f \rangle d\lambda = (\mathcal{F}_0 f_0, \mathcal{F}_\pm f)$$

and hence the Abelian weak wave operators for the pair H_0, H exist and are equal to $\mathcal{F}_\pm^* \mathcal{F}_0$.

The strong wave operators (2.14) also exist. This fact can be deduced from general results of [6] or [16]. Alternatively, Theorem 2.4 entails that the operators \mathbf{G}_0 and \mathbf{G} are H -smooth (as well as H_0 -smooth) in the sense of Kato (see, e.g., [13] or [16]) on all compact intervals $X \subset \Lambda$ which also implies the existence of strong limits (2.14). Finally, we note that under assumption (1.2) the difference $R(z) - R_0(z)$ belongs to the trace class. Therefore the existence and completeness of wave operators (2.14) is a consequence of the Birman-Kreĭn theorem obtained in [5].

The results discussed above can be summarized in the following assertion.

THEOREM 2.7. *Let assumption (1.2) hold. Then wave operators (2.14) exist and satisfy the relation $W_{\pm} = \mathcal{F}_{\pm}^* \mathcal{F}_0$. The operators W_{\pm} are isometric and complete, that is their ranges $\text{Ran } W_{\pm} = \mathcal{H}^{(c)}$. The intertwining property $HW_{\pm} = W_{\pm}H_0$ holds.*

2.3. Since the scattering operator $\mathcal{S} = W_+^* W_-$ commutes with the operator H_0 , the operator

$$(2.22) \quad \mathcal{F}_0 \mathcal{S} \mathcal{F}_0^* = \mathcal{F}_+ \mathcal{F}_-^*$$

acts in the space $L_2(\mathbb{R}_+; \mathbb{C}^2)$ as multiplication by a 2×2 matrix-valued function

$$(2.23) \quad S(\lambda) = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{pmatrix}$$

known as the scattering matrix. According to (2.11) equality (2.22) means that

$$(2.24) \quad S(\lambda) \Gamma_-(\lambda) f = \Gamma_+(\lambda) f, \quad \lambda \in \Lambda, \quad \forall f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}).$$

This equation determines the scattering matrix uniquely. Let us show that its solution is given by the formula

$$(2.25) \quad S(\lambda) = I - 2\pi i \Gamma_0(\lambda)(V - VR(\lambda + i0)V)\Gamma_0^*(\lambda), \quad \lambda \in \Lambda.$$

Let us set $\Gamma_0 = \Gamma_0(\lambda)$, $R_0 = R_0(\lambda + i0)$ and $R = R(\lambda + i0)$. By virtue of definition (2.10), we have to check that

$$(I - 2\pi i \Gamma_0(V - VRV)\Gamma_0^*)\Gamma_0(I - VR^*) = \Gamma_0(I - VR).$$

In view of identity (2.16), it suffices to verify that

$$(I - RV)(R_0 - R_0^*)(I - VR^*) = R - R^*.$$

This equality is a direct consequence of the resolvent identity (2.4). Thus, we have proven the following result.

THEOREM 2.8. *Let assumption (1.2) hold. Then the scattering matrix $S(\lambda)$ for the pair H_0, H admits representation (2.25).*

According to (2.25) the scattering matrix $S(\lambda)$ is a continuous function of $\lambda \in \Lambda$.

Since $v_j(x) = \overline{v_j(x)}$ for $j = 0, 1$, the resolvent $R(z)$ commutes with the complex conjugation which can formally be written in terms of its kernel as $\overline{R(x, y; z)} = R(x, y; \bar{z})$. Taking also into account that $R(\bar{z}) = R^*(z)$, we see that the Green function $R(x, y; z)$ is symmetric, that is

$$(2.26) \quad R(x, y; z) = R(y, x; z).$$

In view of representation (2.25), it follows from this relation that

$$(2.27) \quad s_{11}(\lambda) = s_{22}(\lambda).$$

3. A homogeneous equation

3.1. Let us study a structure of the exceptional set \mathcal{N} . Suppose that, for one of the signs, equation (2.7) is satisfied. Set

$$(3.1) \quad \psi = R_0(\lambda \pm i0)\mathbf{G}^*\mathbf{f}.$$

Then it follows from (2.7) that

$$(3.2) \quad \mathbf{f} + \mathbf{G}_0\psi = 0$$

and hence

$$(3.3) \quad \psi + R_0(\lambda \pm i0)V\psi = 0.$$

Taking into account definition (2.5) of the operator \mathbf{G} , we see that

$$(3.4) \quad V\psi = -\mathbf{G}^*\mathbf{f} = \varphi_0 + D\varphi_1 \quad \text{where} \quad \varphi_j = -\sqrt{|v_j|}f_j \in L_1(\mathbb{R}), \quad j = 0, 1.$$

The functions $R_0(\lambda \pm i0)\varphi_0$ and $R_0(\lambda \pm i0)D\varphi_1$ are well defined by Proposition 2.2.

PROPOSITION 3.1. *Suppose that assumption (1.2) holds. Let $\mathbf{f} \in \mathcal{G}$, $\mathbf{f} \neq 0$, satisfy equation (2.7), and let ψ be defined by formula (3.1). Then ψ is not identically zero, $\psi \in C^2(\mathbb{R})$, the functions $\psi''(x)$ and $\psi'''(x) - v_1(x)\psi'(x)$ are absolutely continuous and differential equation (1.3) is satisfied.*

PROOF. If $\psi = 0$, then $\mathbf{f} = 0$ according to equation (3.2). By virtue of Proposition 2.2, the inclusion $\psi \in C^2(\mathbb{R})$ and the absolute continuity of $\psi''(x)$ follow from equation (3.3) and representation (3.4). For $\theta \in C_0^\infty(\mathbb{R})$, we have that

$$(R_0(\lambda \pm i0)V\psi, (D^4 - \lambda)\theta) = (v_1\psi', R_0(\lambda \mp i0)(D^4 - \lambda)\theta') + (v_0\psi, R_0(\lambda \mp i0)(D^4 - \lambda)\theta).$$

Using equation (3.3) and the fact that $R_0(\lambda \mp i0)(D^4 - \lambda)\theta = \theta$ for all $\theta \in C_0^\infty(\mathbb{R})$, we can rewrite this equality as

$$-(\psi, (D^4 - \lambda)\theta) = (v_1\psi', \theta') + (v_0\psi, \theta).$$

Thus,

$$(-\psi''' + v_1\psi', \theta') = (\lambda\psi - v_0\psi, \theta)$$

where $-\psi''' + v_1\psi' \in L_1^{(loc)}(\mathbb{R})$ and $\lambda\psi - v_0\psi \in L_1^{(loc)}(\mathbb{R})$. It follows that the derivative in the sense of distributions of the function $\psi''' - v_1\psi'$ equals $\lambda\psi - v_0\psi$. This implies that the function $\psi''' - v_1\psi'$ is absolutely continuous and differential equation (1.3) is satisfied. \square

Below differential equation (1.3) is always understood in the sense specified in Proposition 3.1.

Next we find asymptotic behavior of $\psi(x)$ as $|x| \rightarrow \infty$. To that end, we need the following standard assertion.

LEMMA 3.2. *Suppose that, for one of the signs, equation (2.7) is satisfied. Then*

$$(3.5) \quad \Gamma_0(\lambda)\mathbf{G}^*\mathbf{f} = 0.$$

PROOF. It follows from equation (2.7) that

$$\lim_{\varepsilon \rightarrow 0} (\mathbf{f} + \mathbf{G}_0 R_0(\lambda \pm i\varepsilon)\mathbf{G}^*\mathbf{f}, \mathbf{G} R_0(\lambda \pm i\varepsilon)\mathbf{G}^*\mathbf{f}) = 0.$$

Taking here the imaginary part and using that the operator $V = \mathbf{G}^*\mathbf{G}_0$ is symmetric, we obtain the equality

$$\lim_{\varepsilon \rightarrow 0} ((R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon))\mathbf{G}^*\mathbf{f}, \mathbf{G}^*\mathbf{f}) = 0.$$

Now (3.5) is a consequence of relation (2.16). \square

Below integrals containing derivatives of L_1 -functions (for example, $\varphi'_1(x)$) are understood in the sense of distributions, that is integration by parts is tacitly assumed. Using formula (2.8) and representation (3.4), we can reformulate Lemma 3.2 in the following way.

COROLLARY 3.3. *Suppose that, for one of the signs, equation (2.7) is satisfied. Define the function ψ by formula (3.1). Then for both signs “ \pm ”*

$$(3.6) \quad \int_{-\infty}^{\infty} e^{\pm ikx} (V\psi)(x) dx = 0, \quad k = \sqrt[4]{\lambda}.$$

We use also the following simple result.

LEMMA 3.4. *Let $\varphi \in L_1(\mathbb{R})$. Then*

$$(3.7) \quad (R_0(\lambda + i0)\varphi)(x) = \frac{i}{4k^3} e^{\pm ikx} \int_{-\infty}^{\infty} e^{\mp ik y} \varphi(y) dy + o(1)$$

as $x \rightarrow \pm\infty$.

PROOF. Suppose for definiteness that $x \rightarrow +\infty$. According to (2.3) the function $-4ik^3(R_0(\lambda + i0)\varphi)(x)$ consists of two terms. The first of them equals

$$(3.8) \quad \begin{aligned} & \int_{-\infty}^{\infty} e^{ik|x-y|} \varphi(y) dy = e^{ikx} \int_{-\infty}^{\infty} e^{-iky} \varphi(y) dy \\ & - e^{ikx} \int_x^{\infty} e^{-iky} \varphi(y) dy + e^{-ikx} \int_x^{\infty} e^{iky} \varphi(y) dy. \end{aligned}$$

Since $\varphi \in L_1(\mathbb{R})$, both integrals over (x, ∞) in the right-hand side tend to zero as $x \rightarrow +\infty$. The second term equals

$$(3.9) \quad \int_{-\infty}^{\infty} e^{-k|x-y|} \varphi(y) dy = e^{-kx} \int_{-\infty}^x e^{ky} \varphi(y) dy + e^{kx} \int_x^{\infty} e^{-ky} \varphi(y) dy.$$

Since

$$\left| \int_{-\infty}^x e^{ky} \varphi(y) dy \right| \leq e^{kx/2} \int_{-\infty}^{x/2} |\varphi(y)| dy + e^{kx} \int_{x/2}^x |\varphi(y)| dy$$

and

$$\left| \int_x^{\infty} e^{-ky} \varphi(y) dy \right| \leq e^{-kx} \int_x^{\infty} |\varphi(y)| dy,$$

both terms in the right-hand side of (3.9) tend to zero as $x \rightarrow +\infty$. \square

Of course asymptotics of $(R_0(\lambda - i0)\varphi)(x)$ is obtained from (3.7) by the complex conjugation.

In view of Lemma 3.4 equation (3.3) and condition (3.6) imply that

$$(3.10) \quad \lim_{|x| \rightarrow \infty} \psi(x) = 0.$$

Let us formulate the results obtained in the following intermediary assertion.

PROPOSITION 3.5. *Under the assumptions of Proposition 3.1 condition (3.10) is satisfied.*

3.2. In this subsection we use specific methods of ordinary differential equations. Let us first of all rewrite equation (1.3) as a system of four equations of the first order. We set

$$(3.11) \quad \mathbf{u} = (u_1 u_2, u_3, u_4), \quad \text{where} \quad u_1 = u, \quad u_2 = u', \quad u_3 = u'', \quad u_4 = u''' - v_1 u',$$

and

$$(3.12) \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}, \quad K(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & v_1(x) & 0 & 0 \\ -v_0(x) & 0 & 0 & 0 \end{pmatrix}.$$

Then equation (1.3) is equivalent to the system

$$(3.13) \quad \mathbf{u}'(x) = A\mathbf{u}(x) + K(x)\mathbf{u}(x).$$

Clearly, the matrix A has eigenvalues $\gamma_1 = ik$, $\gamma_2 = -ik$, $\gamma_3 = -k$, $\gamma_4 = k$. We denote by $\mathbf{p}_j = (1, \gamma_j, \gamma_j^2, \gamma_j^3)$, $j = 1, 2, 3, 4$, the corresponding eigenvectors. Let (non-orthogonal) projectors P_j be defined by the relation

$$P_j \mathbf{f} = c_j \mathbf{p}_j \quad \text{if} \quad \mathbf{f} = \sum_{l=1}^4 c_l \mathbf{p}_l.$$

Then $P_j^2 = P_j$, $P_j P_l = 0$ if $j \neq l$, $AP_j = \gamma_j P_j$ and

$$I = \sum_{j=1}^4 P_j, \quad e^{Ax} = \sum_{j=1}^4 e^{\gamma_j x} P_j.$$

Although the following result is a particular case of Problem 29, Chapter 3, of [8], we give its proof for a completeness of our presentation.

PROPOSITION 3.6. *Let assumption (1.2) hold. Then, for each of the signs “ \pm ”, system (3.13) has four solutions $\mathbf{u}_j^{(\pm)}(x, \lambda)$, $j = 1, 2, 3, 4$, such that*

$$(3.14) \quad \mathbf{u}_j^{(\pm)}(x, \lambda) = e^{\pm \gamma_j x} (\mathbf{p}_j + o(1))$$

as $x \rightarrow \pm\infty$. Estimates of the remainders in (3.14) are uniform with respect to λ from compact subintervals of \mathbb{R}_+ .

PROOF. We suppose that $x \rightarrow +\infty$ and omit the upper index “ \pm ”. Pick some $j = 1, 2, 3, 4$. Let us set

$$Y_1(x) = Y_2(x) = e^{-kx} P_3, \quad Y_3(x) = 0, \quad Y_4(x) = \sum_{l=1}^3 e^{\gamma_l x} P_l$$

and

$$Z_j(x) = e^{Ax} - Y_j(x).$$

Remark that

$$(3.15) \quad |Y_j(x)| \leq C e^{(\sigma_j - k)x} \quad \text{for} \quad x \geq 0 \quad \text{and} \quad |Z_j(x)| \leq C e^{\sigma_j x} \quad \text{for} \quad x \leq 0$$

where $\sigma_j = \operatorname{Re} \gamma_j$. Let us choose a number a such that

$$(3.16) \quad 2C \int_a^\infty |K(y)| dy \leq 1$$

and consider an integral equation

$$(3.17) \quad \mathbf{u}_j(x) = e^{\gamma_j x} \mathbf{p}_j + \int_a^x Y_j(x-y)K(y)\mathbf{u}_j(y)dy - \int_x^\infty Z_j(x-y)K(y)\mathbf{u}_j(y)dy.$$

Below we also omit the index j . Let us show that equation (3.17) has a solution $\mathbf{u}(x)$ satisfying an estimate

$$(3.18) \quad |\mathbf{u}(x)| \leq 2pe^{\sigma x}, \quad p = \mathbf{p}, \quad x \geq a.$$

We use the method of successive approximations setting $\mathbf{u}^{(0)}(x) = \mathbf{p}e^{\gamma x}$ and

$$(3.19) \quad \mathbf{u}^{(l+1)}(x) = \mathbf{p}e^{\gamma x} + \int_a^x Y(x-y)K(y)\mathbf{u}^{(l)}(y)dy - \int_x^\infty Z(x-y)K(y)\mathbf{u}^{(l)}(y)dy.$$

Let us check that, for all l ,

$$(3.20) \quad |\mathbf{u}^{(l)}(x) - \mathbf{u}^{(l-1)}(x)| \leq 2^{-l}pe^{\sigma x}.$$

Supposing (3.20) for some l and using definition (3.19), we obtain an estimate

$$|\mathbf{u}^{(l+1)}(x) - \mathbf{u}^{(l)}(x)| \leq 2^{-l}p \left(\int_a^x |Y(x-y)||K(y)|e^{\sigma y}dy + \int_x^\infty |Z(x-y)||K(y)|e^{\sigma y}dy \right).$$

By virtue of inequalities (3.15) and condition (3.16) this expression does not exceed

$$2^{-l}p C e^{\sigma x} \int_a^\infty |K(y)|dy \leq 2^{-l-1}pe^{\sigma x}.$$

This proves estimate (3.20) for $l+1$ in place of l and hence for all l . Thus, the sequence $\mathbf{u}^{(l)}(x)$ converges as $l \rightarrow \infty$ to a function $\mathbf{u}(x)$ satisfying estimate (3.18). Passing in (3.19) to the limit $l \rightarrow \infty$, we get equation (3.17).

To prove asymptotics (3.14) for the function $\mathbf{u}(x)$, we combine inequalities (3.15) and (3.18). Obviously, the last integral in the right-hand side of (3.17) is $o(e^{\sigma x})$ as $x \rightarrow \infty$. The first integral is estimated by

$$2C p e^{\sigma x} \left(e^{-kx/2} \int_a^{x/2} |K(y)|dy + \int_{x/2}^x |K(y)|dy \right)$$

which is also $o(e^{\sigma x})$.

Finally, a direct differentiation shows that a solution of integral equation (3.17) satisfies also system (3.13). \square

Let $u_j^{(\pm)}(x, \lambda)$ be the first component of the vector $\mathbf{u}_j^{(\pm)}(x, \lambda)$. Proposition 3.6 can be reformulated in terms of solutions of equation (1.3).

PROPOSITION 3.7. *Let assumption (1.2) hold. Then, for each of the signs “ \pm ”, differential equation (1.3) has four solutions $u_j^{(\pm)}(x, \lambda)$, $j = 1, 2, 3, 4$, such that*

$$(3.21) \quad \begin{aligned} u_1^{(\pm)}(x, \lambda) &= e^{\pm ikx}(1 + o(1)), & u_2^{(\pm)}(x, \lambda) &= e^{\mp ikx}(1 + o(1)), \\ u_3^{(\pm)}(x, \lambda) &= e^{\mp kx}(1 + o(1)), & u_4^{(\pm)}(x, \lambda) &= e^{\pm kx}(1 + o(1)) \end{aligned}$$

as $x \rightarrow \pm\infty$. Estimates of the remainders in (3.21) are uniform with respect to λ from compact subintervals of \mathbb{R}_+ .

REMARK 3.8. It follows from formulas (3.11) that asymptotic relations (3.21) are two times differentiable with respect to x . Moreover,

$$d^3 u_j^{(\pm)}(x, \lambda)/dx^3 - v_1(x) du_j^{(\pm)}(x, \lambda)/dx = \pm \gamma_j^3 e^{\pm \gamma_j x} (\mathbf{p}_j + o(1))$$

as $x \rightarrow \pm\infty$.

Every solution $u(x)$ of equation (1.3) is a linear combination of the solutions $u_j^{(\pm)}(x, \lambda)$, $j = 1, 2, 3, 4$. Therefore if $u(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, then $u(x) = c^{(\pm)}u_3^{(\pm)}(x, \lambda)$ for some constant $c^{(\pm)}$ and hence belongs to $L_2(\mathbb{R}_\pm)$. In particular, we have

PROPOSITION 3.9. *Suppose that a function $u(x)$ satisfies equation (1.3) and $u(x) = o(1)$ as $x \rightarrow \pm\infty$. Then $u(x) = 0$ if λ is not an eigenvalue of the operator H .*

Combining Propositions 3.1, 3.5 and 3.9, we obtain that every $\lambda \in \mathcal{N}$ is necessarily an eigenvalue of the operator H . Taking also into account Theorem 2.7, we see that the singular continuous spectrum of the operator H is empty. Conversely, if ψ is an eigenfunction of the operator H , then it satisfies equation (3.3) and hence \mathbf{f} defined by (3.2) satisfies equation (2.7). Positive eigenvalues of H are of course simple.

Let us finally show that eigenvalues of the operator H might accumulate at the point zero only. Suppose on the contrary that eigenvalues $\lambda_n = k_n^4 \rightarrow \lambda_0 = k_0^4 > 0$. Let ψ_n be the corresponding normalized eigenfunctions. By Proposition 3.7, we have that

$$\psi_n(x) = a_n^{(\pm)} e^{-k_n|x|} (1 + o(1))$$

as $x \rightarrow \pm\infty$. The estimate of the remainder here is uniform with respect to n . Since $\|\psi_n\| = 1$, we have that $|a_n^{(\pm)}| \leq C < \infty$. Therefore, for all $\varepsilon > 0$, and sufficiently large $R = R(\varepsilon)$

$$(3.22) \quad \int_{|x| \geq R} |\psi_n(x)|^2 dx < \varepsilon$$

uniformly with respect to n . Moreover, we have that $h[\psi_n, \psi_n] = \lambda_n$ and hence $\|\psi_n\|_{H^2(\mathbb{R})} \leq C < \infty$ according to estimate (2.1). Together with (3.22), this ensures compactness of the set of the functions ψ_n in the space $L_2(\mathbb{R})$ which contradicts their orthogonality.

Thus, we have obtained

THEOREM 3.10. *Let assumption (1.2) hold. Then \mathcal{N} coincides with the set of positive eigenvalues of the operator H , the singular continuous spectrum of the operator H is empty and eigenvalues of the operator H might accumulate at the point zero only.*

According to Theorem 3.10 the absolutely continuous subspace $\mathcal{H}^{(c)}$ of the operator H equals $\mathcal{H}^{(c)} = \mathcal{H} \ominus \mathcal{H}^{(p)}$ where $\mathcal{H}^{(p)}$ is the subspace spanned by eigenfunctions (of the point spectrum) of the operator H .

4. Eigenfunctions of the continuous spectrum

4.1. Eigenfunctions $\psi_j^{(\pm)}(x, \lambda)$, $j = 1, 2$, of the continuous spectrum of the operator H are defined by the formula

$$(4.1) \quad \psi_j^{(\pm)}(\lambda) = \psi_j^{(0)}(\lambda) - R(\lambda \mp i0)V\psi_j^{(0)}(\lambda), \quad j = 1, 2,$$

where $\psi_1^{(0)}(x, \lambda) = e^{ikx}$, $\psi_2^{(0)}(x, \lambda) = e^{-ikx}$ and $k = \sqrt[4]{\lambda} > 0$. Set $\psi_j(x, \lambda) = \psi_j^{(-)}(x, \lambda)$, $j = 1, 2$. By virtue of property (2.26) we have that $\psi_1^{(+)}(x, \lambda) = \overline{\psi_2(x, \lambda)}$

and $\psi_2^{(+)}(x, \lambda) = \overline{\psi_1(x, \lambda)}$. The functions $\psi_j(x, \lambda)$ are known also as wave functions. The resolvent identity (2.4) implies that the functions $\psi_j^{(\pm)}(x, \lambda)$ satisfy the Lippmann-Schwinger equation

$$(4.2) \quad \psi_j^{(\pm)}(\lambda) = \psi_j^{(0)}(\lambda) - R_0(\lambda \mp i0)V\psi_j^{(\pm)}(\lambda), \quad j = 1, 2.$$

Remark that the right-hand side here is correctly defined. Indeed, it follows from Theorem 2.4 that $g\psi_j(\lambda), g\psi_j'(\lambda) \in L_2(\mathbb{R})$ for an arbitrary function $g \in L_2(\mathbb{R})$ and hence the functions $V\psi_j(\lambda)$ admit representation (3.4).

Similarly to the proof of Proposition 3.1, it is easy to deduce from (4.2) that the wave functions satisfy also differential equation (1.3). Their asymptotics as $|x| \rightarrow \infty$ can be found with a help of Lemma 3.4. Thus, we obtain the following result.

PROPOSITION 4.1. *Let assumption (1.2) hold. Suppose that $\lambda = k^4$ is not an eigenvalue of the operator H . Let matrix (2.23) be defined by equation (2.25), and let the solutions $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$ of equation (1.3) be defined by formula (4.1). Then the asymptotic relations*

$$(4.3) \quad \begin{cases} \psi_1(x, \lambda) = e^{ikx} + s_{21}(\lambda)e^{-ikx} + o(1), & x \rightarrow -\infty, \\ \psi_1(x, \lambda) = s_{11}(\lambda)e^{ikx} + o(1), & x \rightarrow \infty, \end{cases}$$

and

$$(4.4) \quad \begin{cases} \psi_2(x, \lambda) = e^{-ikx} + s_{12}(\lambda)e^{ikx} + o(1), & x \rightarrow \infty, \\ \psi_2(x, \lambda) = s_{22}(\lambda)e^{-ikx} + o(1), & x \rightarrow -\infty, \end{cases}$$

hold.

REMARK 4.2. Here and below all asymptotic relations are differentiable in the sense of Remark 3.8. Actually, for example, the first relation (4.3) entails, by virtue of Proposition 3.7, that

$$\psi_1(x, \lambda) = s_{21}(\lambda)u_1^{(-)}(x, \lambda) + u_2^{(-)}(x, \lambda),$$

where the functions $u_1^{(-)}$ and $u_2^{(-)}$ are differentiable according to Remark 3.8.

Theorem 2.6 can be reformulated as an expansion of an arbitrary function in a generalized Fourier integral over the eigenfunctions $\psi_1^{(\pm)}(x, \lambda)$ and $\psi_2^{(\pm)}(x, \lambda)$ of the operator H . Indeed, according to definition (4.1) for an arbitrary $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$

$$((I - VR(\lambda \pm i0))f, \psi_j^{(0)}(\lambda)) = (f, \psi_j^{(\pm)}(\lambda)), \quad j = 1, 2,$$

and according to definitions (2.9) and (2.10)

$$(4.5) \quad \Gamma_{\pm}(\lambda)f = 2^{-1}(2\pi)^{-1/2}\lambda^{-3/8}((f, \psi_1^{(\pm)}(\lambda)), (f, \psi_2^{(\pm)}(\lambda))).$$

Therefore formula (2.11) reads as $(\mathcal{F}_{\pm}f)(\lambda) = (\tilde{f}_1(\lambda), \tilde{f}_2(\lambda))$ where

$$\tilde{f}_j(\lambda) = 2^{-1}(2\pi)^{-1/2}\lambda^{-3/8} \int_{-\infty}^{\infty} \overline{\psi_j^{(\pm)}(x, \lambda)} f(x) dx.$$

It follows that the relation $P^{(c)}f = \mathcal{F}_{\pm}^* \mathcal{F}_{\pm} f$ can (formally) be written as

$$(P^{(c)}f)(x) = 2^{-1}(2\pi)^{-1/2} \sum_{j=1}^2 \int_0^{\infty} \psi_j^{(\pm)}(x, \lambda) \tilde{f}_j(\lambda) \lambda^{-3/8} d\lambda.$$

By virtue of representation (4.5), equality (2.24) is equivalent to relations

$$(4.6) \quad \begin{cases} s_{11}(\lambda)\overline{\psi_1(x, \lambda)} + s_{12}(\lambda)\overline{\psi_2(x, \lambda)} &= \psi_2(x, \lambda), \\ s_{21}(\lambda)\psi_1(x, \lambda) + s_{22}(\lambda)\psi_2(x, \lambda) &= \psi_1(x, \lambda). \end{cases}$$

4.2. Let us show that asymptotics (4.3) or (4.4) determine uniquely solutions of equation (1.3). We start with an auxiliary assertion which is true without assumption (1.2).

LEMMA 4.3. *Suppose that the functions v_0 and v_1 are real. Set*

$$(4.7) \quad F_u(r) = (u'''(r) - v_1(r)u'(r))\overline{u(r)} - u''(r)\overline{u'(r)}.$$

Then, for an arbitrary solution $u(x)$ of equation (1.3) and all r , we have

$$(4.8) \quad \operatorname{Im} F_u(r) = \operatorname{Im} F_u(-r).$$

PROOF. It follows from equation (1.3) that

$$\operatorname{Im} \int_{-r}^r (u''' - v_1 u')' \overline{u} dx = 0.$$

Integrating here by parts, we get equality (4.8). \square

Now we can formulate the uniqueness result.

PROPOSITION 4.4. *Let assumption (1.2) hold. Suppose that $\lambda = k^4$ is not an eigenvalue of the operator H . If a solution $u(x)$ of equation (1.3) satisfies the conditions*

$$(4.9) \quad \begin{cases} u(x) = \sigma_+ e^{\pm i k x} + o(1), & x \rightarrow \infty, \\ u(x) = \sigma_- e^{\mp i k x} + o(1), & x \rightarrow -\infty, \end{cases}$$

for one of the signs and some numbers σ_+ and σ_- , then $\sigma_+ = \sigma_- = 0$ and $u(x) = 0$.

PROOF. It follows (see Remark 4.2) from relations (4.9) that function (4.7) has asymptotics

$$F_u(r) = 2(\pm i k)^3 |\sigma_+|^2 + o(1), \quad r \rightarrow \infty,$$

and

$$F_u(-r) = 2(\mp i k)^3 |\sigma_-|^2 + o(1), \quad r \rightarrow -\infty.$$

Using Lemma 4.3, we find that $|\sigma_+|^2 + |\sigma_-|^2 = 0$ and hence $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Thus, $u(x) = 0$ by Proposition 3.9. \square

COROLLARY 4.5. *If a solution $\tilde{\psi}_1(x)$ of equation (1.3) has asymptotics (4.3) with some coefficients s_{11} and s_{21} , then necessarily s_{11} and s_{21} are the entries of the scattering matrix and $\tilde{\psi}_1 = \psi_1$. Similarly, if a solution $\tilde{\psi}_2(x)$ of equation (1.3) has asymptotics (4.4) with some coefficients s_{12} and s_{22} , then necessarily s_{12} and s_{22} are the entries of the scattering matrix and $\tilde{\psi}_2 = \psi_2$.*

Formulas (4.3) and (4.4) give us the definition of the scattering matrix in terms of solutions of differential equation (1.3). Similarly to the Schrödinger equation, the numbers $s_{11}(\lambda)$, $s_{22}(\lambda)$ can be interpreted as transmission coefficients and $s_{21}(\lambda)$, $s_{12}(\lambda)$ can be interpreted as reflection coefficients for a plane wave coming from minus or plus infinity and interacting with the potentials $v_0(x)$ and $v_1(x)$.

4.3. Here we find asymptotics of the wave functions up to terms decaying super-exponentially at infinity. We say that a function $\varphi(x)$ decays super-exponentially if

$$(4.10) \quad \varphi(x) = O(e^{-\varkappa|x|}), \quad \forall \varkappa > 0, \quad |x| \rightarrow \infty.$$

The following result supplements Lemma 3.4.

LEMMA 4.6. *Let a function φ satisfy condition (4.10). Then*

$$\begin{aligned} (R_0(\lambda + i0)\varphi)(x) &= \frac{i}{4k^3} e^{\pm ikx} \int_{-\infty}^{\infty} e^{\mp iky} \varphi(y) dy \\ &- \frac{1}{4k^3} e^{-k|x|} \int_{-\infty}^{\infty} e^{\pm ky} \varphi(y) dy + O(e^{-\varkappa|x|}), \quad \forall \varkappa > 0, \end{aligned}$$

as $x \rightarrow \pm\infty$.

PROOF. Suppose again for definiteness that $x \rightarrow +\infty$. According to (2.3) the function $-4ik^3(R_0(\lambda + i0)\varphi)(x)$ consists of the terms (3.8) and

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-k|x-y|} \varphi(y) dy &= e^{-kx} \int_{-\infty}^{\infty} e^{ky} \varphi(y) dy \\ (4.11) \quad &- e^{-kx} \int_x^{\infty} e^{ky} \varphi(y) dy + e^{kx} \int_x^{\infty} e^{-ky} \varphi(y) dy. \end{aligned}$$

The integrals over (x, ∞) in the right-hand sides of (3.8) and (4.11) decay super-exponentially by virtue of condition (4.10). \square

Let us return to the Lippmann-Schwinger equation (4.2) and take into account that the functions $\psi_j(x, \lambda)$ and $\psi'_j(x, \lambda)$, $j = 1, 2$, are bounded. Lemma 4.6 yields now a more precise form of Proposition 4.1.

PROPOSITION 4.7. *Let the functions v_0 and v_1 satisfy condition (4.10). Suppose that $\lambda = k^4$ is not an eigenvalue of the operator H . Let matrix (2.23) be defined by equation (2.25), and let the solutions $\psi_1(x, \lambda)$ and $\psi_2(x, \lambda)$ of equation (1.3) be defined by formula (4.1). Then we have the asymptotic relations*

$$(4.12) \quad \begin{cases} \psi_1(x, \lambda) = s_{11}(\lambda) e^{ikx} + b_{11}(\lambda) e^{-kx} + O(e^{-\varkappa x}), & x \rightarrow \infty, \\ \psi_1(x, \lambda) = e^{ikx} + s_{21}(\lambda) e^{-ikx} + b_{21}(\lambda) e^{kx} + O(e^{\varkappa x}), & x \rightarrow -\infty, \end{cases}$$

and

$$(4.13) \quad \begin{cases} \psi_2(x, \lambda) = e^{-ikx} + s_{12}(\lambda) e^{ikx} + b_{12}(\lambda) e^{-kx} + O(e^{-\varkappa x}), & x \rightarrow \infty, \\ \psi_2(x, \lambda) = s_{22}(\lambda) e^{-ikx} + b_{22}(\lambda) e^{kx} + O(e^{\varkappa x}), & x \rightarrow -\infty, \end{cases}$$

where \varkappa is arbitrary and

$$b_{1l}(\lambda) = \frac{1}{4k^3} \int_{-\infty}^{\infty} e^{ky} (V\psi_l(\lambda))(y) dy, \quad b_{2l}(\lambda) = \frac{1}{4k^3} \int_{-\infty}^{\infty} e^{-ky} (V\psi_l(\lambda))(y) dy,$$

$l = 1, 2$. The asymptotic coefficients $b_{jl}(\lambda)$, $j, l = 1, 2$, are continuous functions of $\lambda \in \Lambda$.

Let us find a relation between the sets of coefficients $s_{jl}(\lambda)$ and $b_{jl}(\lambda)$. To that end, we plug asymptotics (4.12) and (4.13) into system (4.6). Neglecting super-exponentially decaying terms, we have that, as $x \rightarrow \infty$,

$$(4.14) \quad s_{11}(\overline{s_{11}} e^{-ikx} + \overline{b_{11}} e^{-kx}) + s_{12}(\overline{s_{12}} e^{-ikx} + \overline{b_{12}} e^{-kx}) = e^{-ikx} + b_{12} e^{-kx}$$

and

$$(4.15) \quad \begin{aligned} s_{21}(\overline{s_{11}}e^{-ikx} + \overline{b_{11}}e^{-kx}) + s_{22}(e^{ikx} + \overline{s_{12}}e^{-ikx} + \overline{b_{12}}e^{-kx}) \\ = s_{11}e^{ikx} + b_{11}e^{-kx}. \end{aligned}$$

Similarly, if $x \rightarrow -\infty$, we have that

$$(4.16) \quad \begin{aligned} s_{11}(e^{-ikx} + \overline{s_{21}}e^{ikx} + \overline{b_{21}}e^{kx}) + s_{12}(\overline{s_{22}}e^{ikx} + \overline{b_{22}}e^{kx}) \\ = s_{22}e^{-ikx} + b_{22}e^{kx} \end{aligned}$$

and

$$(4.17) \quad s_{21}(\overline{s_{21}}e^{ikx} + \overline{b_{21}}e^{kx}) + s_{22}(\overline{s_{22}}e^{ikx} + \overline{b_{22}}e^{kx}) = e^{ikx} + b_{21}e^{kx}.$$

Comparing the coefficients at e^{-ikx} in the left- and right-hand sides of equations (4.14) and (4.15), we find that

$$(4.18) \quad |s_{11}|^2 + |s_{12}|^2 = 1$$

and

$$(4.19) \quad s_{21}\overline{s_{11}} + s_{22}\overline{s_{12}} = 0.$$

Comparing the coefficients at e^{ikx} in the left- and right-hand sides of equations (4.16) and (4.17), we obtain again relations (4.19) and

$$(4.20) \quad |s_{21}|^2 + |s_{22}|^2 = 1.$$

Identities (4.18), (4.19) and (4.20) show that the scattering matrix is a unitary operator in \mathbb{C}^2 . This result has already been obtained in Section 2 as a consequence of the completeness of the wave operators.

Comparing the coefficients at e^{ikx} (or at e^{-ikx}) in the left- and right-hand sides of equation (4.15) (or of equation (4.16)), we recover relation (2.27), which has already been obtained in Section 2 as a consequence of the stationary representation (2.25) and of relation (2.26).

Finally, comparing the coefficients at exponentially decreasing terms in equations (4.14) - (4.17), we obtain four relations between the coefficients s_{jl} and b_{jl} . We write them in the matrix form as

$$(4.21) \quad \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{pmatrix} \begin{pmatrix} \overline{b_{11}}(\lambda) & \overline{b_{21}}(\lambda) \\ \overline{b_{12}}(\lambda) & \overline{b_{22}}(\lambda) \end{pmatrix} = \begin{pmatrix} b_{12}(\lambda) & b_{22}(\lambda) \\ b_{11}(\lambda) & b_{21}(\lambda) \end{pmatrix}.$$

We formulate this result in a following assertion.

THEOREM 4.8. *Let the functions v_0 and v_1 satisfy condition (4.10). Suppose that $\lambda = k^4$ is not an eigenvalue of the operator H . Then the asymptotic coefficients s_{jl} and b_{jl} in (4.12) and (4.13) are linked by relation (4.21).*

We emphasize that relations (4.21) as well as (2.27) are consequences of the invariance of the problem with respect to the complex conjugation.

4.4. The approach developed for the problem on the whole line works of course for a similar problem in the space $L_2(\mathbb{R}_+)$. In this case we have to add a boundary condition at the point $x = 0$ which we choose as

$$(4.22) \quad u(0) = u'(0) = 0.$$

The operator H is defined by differential expression (1.1) on functions satisfying (4.22). As a “free” operator, we take $H_0 = D^4$ with the same boundary condition. Since the operator H has a simple spectrum, there is now only one wave function

$\psi(x, \lambda)$ and the scattering operator acts in the space $L_2(\mathbb{R}_+)$ as multiplication by the function (scattering matrix) $s(\lambda)$.

Let us construct the function $\psi(x, \lambda)$. Remark first that the function

$$(4.23) \quad \psi_0(x, \lambda) = \cos(kx + \pi/4) - 2^{-1/2}e^{-kx}$$

satisfies the equation $u^{(4)}(x) = \lambda u(x)$ and boundary condition (4.22). Assume as usual that λ is not an eigenvalue of H . A solution $\psi(x, \lambda)$ of equation (1.3) is then defined (cf. (4.1)) by the formula $\psi(\lambda) = \psi_0(\lambda) - R(\lambda + i0)V\psi_0(\lambda)$. Similarly to Proposition 4.1, it can be shown under short-range assumption (1.2) that

$$(4.24) \quad \psi(x, \lambda) = 2^{-1}(s(\lambda)e^{ikx+\pi i/4} + e^{-ikx-\pi i/4}) + o(1)$$

as $x \rightarrow \infty$. Then we use (cf. Lemma 4.3) that $\text{Im } F_u(r) = 0$ for all solutions $u(x)$ of equation (1.3) satisfying (4.22). Applying this result to the difference of two functions obeying (4.24), we see that actually condition (4.24) distinguishes a unique solution of (1.3). In particular,

$$(4.25) \quad \overline{\psi(x, \lambda)} = s(\lambda)\psi(x, \lambda).$$

Moreover, the identity $\text{Im } F_\psi(r) = 0$ implies that $|s(\lambda)| = 1$.

If the functions $v_1(x)$ and $v_0(x)$ decay super-exponentially, then the remainder $o(1)$ in (4.24) can be replaced by a more precise term $-2^{-1/2}b(\lambda)e^{-kx} + O(e^{-\varkappa x})$ where \varkappa is an arbitrary number. Relation (4.25) entails that

$$(4.26) \quad s(\lambda)\overline{b(\lambda)} = b(\lambda)$$

which plays the role of identity (4.21).

5. Perturbation by a boundary condition

Here we discuss the Hamiltonian $H = D^4$ in the space $L_2(\mathbb{R}_+)$ with some self-adjoint boundary conditions at the point $x = 0$ and calculate explicitly its resolvent. In contrast to previous sections, we avoid here references to general results of scattering theory and give direct proofs of all assertions.

5.1. Self-adjoint extensions of a symmetric operator D^4 with domain $C_0^\infty(\mathbb{R}_+)$ in the space $L_2(\mathbb{R}_+)$ are defined by the formula $(Hu)(x) = u^{(4)}(x)$ on functions $u(x)$ from the Sobolev space $H^4(\mathbb{R}_+)$ satisfying some boundary conditions at the point $x = 0$. Let us describe all of them. “Generic” self-adjoint boundary conditions have the form

$$(5.1) \quad \begin{cases} u''(0) &= \alpha u(0) + \alpha_1 u'(0) \\ u'''(0) &= -\alpha_2 u(0) - \bar{\alpha} u'(0), \end{cases}$$

where α_1 and α_2 are arbitrary real numbers and α is an arbitrary complex number. This family depends on four real constants. We introduce also “exceptional” three-parameters

$$(5.2) \quad u'(0) = \alpha u(0), \quad -u'''(0) + \bar{\alpha} u''(0) = \alpha_2 u(0), \quad \alpha \in \mathbb{C}, \quad \alpha_2 \in \mathbb{R},$$

and a one-parameter

$$(5.3) \quad u(0) = 0, \quad u''(0) = \alpha_1 u'(0), \quad \alpha_1 \in \mathbb{R},$$

families of boundary conditions. To exhaust all self-adjoint extensions, we have to add boundary condition (4.22). The operator D^4 with this boundary condition

will be denoted H_0 . We use also a special notation H_{00} for the operator D^4 with boundary condition (5.3) where $\alpha_1 = 0$, that is

$$(5.4) \quad u(0) = 0, \quad u''(0) = 0.$$

The quadratic form of the operator $H = H(\alpha, \alpha_1, \alpha_2)$ with boundary conditions (5.1) is given by the expression

$$(5.5) \quad h[u, u] = \int_0^\infty |u''(x)|^2 dx + \alpha_2 |u(0)|^2 + 2 \operatorname{Re}(\alpha u(0) \bar{u}'(0)) + \alpha_1 |u'(0)|^2.$$

It is closed on the set $H^2(\mathbb{R}_+)$. In case (5.2) the form is defined by the expression

$$(5.6) \quad h[u, u] = \int_0^\infty |u''(x)|^2 dx + \alpha_2 |u(0)|^2$$

on functions from $H^2(\mathbb{R}_+)$ satisfying the condition $u'(0) = \alpha u(0)$. In case (5.3) expression (5.5) where $\alpha = \alpha_2 = 0$ remains true if the form is restricted on functions $u \in H^2(\mathbb{R}_+)$ satisfying the condition $u(0) = 0$. Finally, the quadratic form of the operator H_0 is given by expression (5.6) where $\alpha_2 = 0$ on functions $u \in H^2(\mathbb{R}_+)$ such that $u(0) = u'(0) = 0$.

Note that $H \leq H_0$ for all boundary conditions because the quadratic form of the operator H_0 is defined on the smallest possible set. Similarly, $H \leq H_{00}$ for boundary conditions (5.1) and all $\alpha_1, \alpha_2, \alpha$ because the quadratic form of the operator H_{00} is defined on a smaller (the boundary condition $u(0) = 0$ is added) set than that of H .

As a “free” operator, it is natural to take the operator H_0 (the same operator as in subs. 4.4). However technically it is more convenient to work with the “intermediary free” operator H_{00} . The reason for this choice of the free operator is that H_{00} is the square of the operator D^2 with the boundary condition $u(0) = 0$.

We start with a construction of the resolvent $R(z)$ of the operator H . As a preliminary step, we shall find an explicit expression (cf. Lemma 2.1) for the resolvent $R_{00}(z)$ of the operator H_{00} .

LEMMA 5.1. *Let $z \in \mathbb{C} \setminus [0, \infty)$ and $\zeta^4 = z$, $\arg \zeta \in (0, \pi/2)$. Then*

$$(5.7) \quad (R_{00}(z)f)(x) = \frac{1}{4\zeta^3} \int_0^\infty (ie^{i\zeta|x-y|} - e^{-\zeta|x-y|} - ie^{i\zeta(x+y)} + e^{-\zeta(x+y)})f(y)dy.$$

PROOF. Using the Fourier sine transform, we see that

$$(R_{00}(z)f)(x) = 2\pi^{-1} \int_0^\infty dk \sin(kx)(k^4 - z)^{-1} \int_0^\infty dy \sin(ky)f(y).$$

Therefore the resolvent kernel equals

$$R_{00}(x, y; z) = -(2\pi)^{-1} \int_{-\infty}^\infty e^{ik(x+y)}(k^4 - z)^{-1} dk + (2\pi)^{-1} \int_{-\infty}^\infty e^{ik|x-y|}(k^4 - z)^{-1} dk.$$

Both integrals can be complemented in the upper half-plane by a big half-circle and then calculated by residues at the points ζ and $i\zeta$. \square

The solutions from $L_2(\mathbb{R}_+)$ of the equation

$$u^{(4)}(x) = zu(x) + f(x)$$

satisfying different boundary conditions at the point $x = 0$ differ by a linear combination of the solutions $e^{i\zeta x}$ and $e^{-\zeta x}$ of the homogeneous equation $u^{(4)}(x) = zu(x)$. Therefore we seek the resolvent $R(z)$ of H in the form (known as the Krein formula)

$$(5.8) \quad R(z) = R_{00}(z) + P(z),$$

where $P(z)$ is a two-dimensional operator defined by equalities

$$(5.9) \quad \begin{aligned} (P(z)f)(x) &= (p_{11}(\zeta)e^{i\zeta x} + p_{12}(\zeta)e^{-\zeta x})Q_+(z)f \\ &+ (p_{21}(\zeta)e^{i\zeta x} + p_{22}(\zeta)e^{-\zeta x})Q_-(z)f \end{aligned}$$

and

$$(5.10) \quad Q_+(z)f = \int_0^\infty e^{i\zeta y} f(y) dy, \quad Q_-(z)f = \int_0^\infty e^{-\zeta y} f(y) dy.$$

Calculating derivatives of expressions (5.7) and (5.9) at $x = 0$, we find that

$$(5.11) \quad \begin{cases} (R(z)f)(0) &= (p_{11}(\zeta) + p_{12}(\zeta))Q_+(z)f + (p_{21}(\zeta) + p_{22}(\zeta))Q_-(z)f, \\ (R(z)f)'(0) &= 2^{-1}\zeta^{-2}(Q_+(z)f - Q_-(z)f) \\ &+ \zeta((ip_{11}(\zeta) - p_{12}(\zeta))Q_+(z)f + (ip_{21}(\zeta) - p_{22}(\zeta))Q_-(z)f), \\ (R(z)f)''(0) &= \zeta^2((-p_{11}(\zeta) + p_{12}(\zeta))Q_+(z)f \\ &+ (-p_{21}(\zeta) + p_{22}(\zeta))Q_-(z)f), \\ (R(z)f)'''(0) &= -2^{-1}(Q_+(z)f + Q_-(z)f) \\ &- \zeta^3((ip_{11}(\zeta) + p_{12}(\zeta))Q_+(z)f + (ip_{21}(\zeta) + p_{22}(\zeta))Q_-(z)f). \end{cases}$$

The coefficients $p_{jl}(\zeta)$ in (5.9) are determined by boundary conditions on the functions $u(x) = (R(z)f)(x)$ at the point $x = 0$. We consider only generic boundary conditions (5.1) although formulas obtained above allow one to treat also easily cases (5.2) and (5.3). Plugging expressions (5.11) into (5.1) and equating coefficients at Q_+f and Q_-f , we obtain two systems of equations for p_{11} , p_{12} and for p_{21} , p_{22} :

$$(5.12) \quad \begin{cases} a_{11}(\zeta)p_{j1}(\zeta) + a_{12}(\zeta)p_{j2}(\zeta) = q_{j1}(\zeta), \\ a_{21}(\zeta)p_{j1}(\zeta) + a_{22}(\zeta)p_{j2}(\zeta) = q_{j2}(\zeta), \end{cases}$$

where $j = 1, 2$. The coefficients of these systems are given by explicit formulas

$$(5.13) \quad \begin{cases} a_{11}(\zeta) = \alpha + i\alpha_1\zeta + \zeta^2, & a_{12}(\zeta) = \alpha - \alpha_1\zeta - \zeta^2, \\ a_{21}(\zeta) = -\alpha_2 - i\bar{\alpha}\zeta + i\zeta^3, & a_{22}(\zeta) = -\alpha_2 + \bar{\alpha}\zeta + \zeta^3, \end{cases}$$

and

$$\begin{cases} q_{11}(\zeta) = -(2\zeta^2)^{-1}\alpha_1, & q_{12}(\zeta) = (2\zeta^2)^{-1}\bar{\alpha} - 2^{-1}, \\ q_{21}(\zeta) = (2\zeta^2)^{-1}\alpha_1, & q_{22}(\zeta) = -(2\zeta^2)^{-1}\bar{\alpha} - 2^{-1}. \end{cases}$$

Of course, system (5.12) can easily be solved. Let us set $\alpha_0 = \alpha_1\alpha_2 - |\alpha|^2$ and

$$(5.14) \quad \begin{aligned} \Omega(\zeta) &= -2^{-1/2}e^{-\pi i/4}\zeta^{-1}(a_{11}(\zeta)a_{22}(\zeta) - a_{12}(\zeta)a_{21}(\zeta)) \\ &= \alpha_0 + (1-i)\alpha_2\zeta + 2i\operatorname{Re}\alpha\zeta^2 - (1+i)\alpha_1\zeta^3 - \zeta^4. \end{aligned}$$

Then

$$(5.15) \quad p_{jl}(\zeta) = 2^{-3/2}e^{-\pi i/4}\zeta^{-3}\Omega(\zeta)^{-1}\mathbf{p}_{jl}(\zeta), \quad j, l = 1, 2,$$

where

$$(5.16) \quad \begin{cases} \mathbf{p}_{11}(\zeta) = -\alpha_0 - 2\operatorname{Re}\alpha\zeta^2 + 2\alpha_1\zeta^3 + \zeta^4, \\ \mathbf{p}_{12}(\zeta) = \alpha_0 + 2i\operatorname{Im}\alpha\zeta^2 + \zeta^4, \\ \mathbf{p}_{21}(\zeta) = \alpha_0 - 2i\operatorname{Im}\alpha\zeta^2 + \zeta^4, \\ \mathbf{p}_{22}(\zeta) = -\alpha_0 + 2\operatorname{Re}\alpha\zeta^2 + 2i\alpha_1\zeta^3 + \zeta^4. \end{cases}$$

Note that

$$\overline{\Omega(\zeta)} = \Omega(i\bar{\zeta}), \quad \zeta \in \mathbb{C},$$

so that zeros of the function $\Omega(\zeta)$ are symmetric with respect to the line $\operatorname{Re} \zeta = \operatorname{Im} \zeta$.

Let us summarize the results obtained.

THEOREM 5.2. *Let $z \in \mathbb{C} \setminus [0, \infty)$ and $\zeta^4 = z$, $\arg \zeta \in (0, \pi/2)$. Then the resolvent $R(z)$ of the operator $H = d^4/dx^4$ with boundary conditions (5.1), (5.2) or (5.3) is given by formula (5.8) where the resolvent $R_{00}(z)$ of the operator H_{00} is determined by formula (5.7) and the operator $P(z)$ is determined by formulas (5.9) and (5.10). In case (5.1) the coefficients $p_{jl}(\zeta)$ are defined by formulas (5.14), (5.15) and (5.16).*

Note a particular case of this result. If $\alpha = \alpha_1 = \alpha_2 = 0$, then $\Omega(\zeta) = -\zeta^4$ and

$$(5.17) \quad P(x, y; z) = -2^{-3/2} e^{-\pi i/4} \zeta^{-3} (e^{i\zeta x} + e^{-\zeta x}) (e^{i\zeta y} + e^{-\zeta y}).$$

Of course explicit formulas for the coefficients $p_{jl}(\zeta)$ in (5.9) can easily be written down also for other boundary conditions. For example, let us consider the operators H_0 and H_1 determined by boundary conditions (4.22) and

$$(5.18) \quad u'(0) = u'''(0) = 0,$$

respectively. Using formulas (5.11), we see that their resolvents are given by the equality

$$(5.19) \quad R_j(z) = R_{00}(z) + P_j(z), \quad j = 0, 1,$$

where $P_j(z)$ are integral operators with kernels

$$(5.20) \quad P_0(x, y; z) = (4\zeta^3)^{-1} (i - 1) (e^{i\zeta x} - e^{-\zeta x}) (e^{i\zeta y} - e^{-\zeta y})$$

and

$$(5.21) \quad P_1(x, y; z) = (2\zeta^3)^{-1} (i e^{i\zeta(x+y)} - e^{-\zeta(x+y)}).$$

5.2. Let us discuss eigenvalues of the operator H corresponding to boundary conditions (5.1). Recall that the function Ω was defined by formula (5.14).

PROPOSITION 5.3. *Zeros of the function $\Omega(\zeta)$ from the sector $\arg \zeta \in (0, \pi/2)$ lie on the ray $\arg \zeta = \pi/4$. A point $\lambda = -k^4$, $k > 0$, is an eigenvalue of the operator H if and only if $\Omega(e^{i\pi/4}k) = 0$, that is*

$$k^4 + \sqrt{2}\alpha_1 k^3 - 2\operatorname{Re} \alpha k^2 + \sqrt{2}\alpha_2 k + \alpha_0 = 0.$$

Multiplicities of an eigenvalue λ of the operator H and of the zero $e^{i\pi/4}k$ of the function Ω coincide. A point $\lambda_0 = -k_0^4$ is a degenerate (of multiplicity 2) eigenvalue of the operator H if and only if

$$(5.22) \quad \alpha_1 = -\sqrt{2}k_0, \quad \alpha = -k_0^2, \quad \alpha_2 = -\sqrt{2}k_0^3.$$

In this case the eigenfunctions are defined by equations

$$\psi_1(x) = \exp((-1 + i)k_0 x / \sqrt{2}), \quad \psi_2(x) = \exp((-1 - i)k_0 x / \sqrt{2})$$

and necessarily

$$(5.23) \quad \Omega(\zeta) = -(\zeta - e^{i\pi/4}k_0)^2 (\zeta^2 + ik_0^2).$$

PROOF. Clearly, a solution

$$(5.24) \quad u(x) = p_1 e^{i\zeta x} + p_2 e^{-\zeta x}$$

of the equation

$$(5.25) \quad u^{(4)}(x) = \zeta^4 u(x)$$

satisfies boundary conditions (5.1) if and only if (cf. (5.12))

$$(5.26) \quad \begin{cases} a_{11}(\zeta)p_1 + a_{12}(\zeta)p_2 = 0, \\ a_{21}(\zeta)p_1 + a_{22}(\zeta)p_2 = 0, \end{cases}$$

where the coefficients $a_{jl}(\zeta)$ are defined by formulas (5.13). Since both functions $e^{i\zeta x}$ and $e^{-\zeta x}$ belong to $L_2(\mathbb{R}_+)$, the point $\lambda = \zeta^4$ is an eigenvalue of the operator H if and only if system (5.26) has a non-trivial solution, that is $\Omega(\zeta) = 0$. It follows that $\arg \zeta = \pi/4$ because necessarily $\lambda < 0$.

A point $\lambda_0 = -k_0^4$ is an eigenvalue of multiplicity 2 if and only if system (5.26) where $\zeta_0 = e^{\pi i/4} k_0$ is satisfied for all numbers p_1 and p_2 , that is

$$(5.27) \quad a_{11}(\zeta_0) = a_{12}(\zeta_0) = 0, \quad a_{21}(\zeta_0) = a_{22}(\zeta_0) = 0.$$

Using (5.13) and solving the first two equations (5.27), we obtain expressions (5.22) for α_1 and α . Similarly, considering the last two equations (5.27), we obtain expression (5.22) for α_2 . Plugging expressions (5.22) into (5.14), we get representation (5.23) so that the point ζ_0 is a double zero of the function $\Omega(\zeta)$.

Conversely, if $\Omega(\zeta_0) = \Omega'(\zeta_0) = 0$, then $\mathbf{p}_{jl}(\zeta_0) = 0$ for all $j, l = 1, 2$ because the functions $p_{jl}(\zeta)$ can have only simple poles. This is a consequence of representations (5.8), (5.9) for the resolvent $R(z)$ whose poles are simple. By virtue of (5.16) the equations $\mathbf{p}_{12}(\zeta_0) = \mathbf{p}_{21}(\zeta_0) = 0$ yield $\text{Im } \alpha = 0$, $\alpha_0 + \zeta_0^4 = 0$. Therefore the equations $\mathbf{p}_{11}(\zeta_0) = \mathbf{p}_{22}(\zeta_0) = 0$ can be written as

$$\begin{cases} -\alpha + \alpha_1 \zeta_0 + \zeta_0^2 = 0, \\ \alpha + i\alpha_1 \zeta_0 + \zeta_0^2 = 0, \end{cases}$$

whence $(1+i)\alpha_1 + 2\zeta_0 = 0$ and $(1-i)\alpha_1 \zeta_0 = 2\alpha$. This implies equations (5.22) so that equations (5.27) for numbers (5.13) are also satisfied. Thus, function (5.24) where $\zeta = \zeta_0$ satisfies boundary conditions (5.1) for all p_1, p_2 and hence $-k_0^4$ is an eigenvalue of multiplicity 2. \square

Since the rank of the operator $R(z) - R_{00}(z)$ equals 2 and $H_{00} \geq 0$, the operator H might have at most 2 negative eigenvalues with multiplicity taken into account. The following result (cf. [17]) makes this assertion more precise.

PROPOSITION 5.4. *The total numbers of negative eigenvalues (counted with their multiplicity) of the operators H and*

$$A = \begin{pmatrix} \alpha_2 & \bar{\alpha} \\ \alpha & \alpha_1 \end{pmatrix} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

coincide.

PROOF. Let $E(\mathbb{R}_-)$ and $\mathcal{E}(\mathbb{R}_-)$ be spectral projectors of the operators H and A , respectively, corresponding to the set $\mathbb{R}_- = (-\infty, 0)$. We have to check that

$$N := \dim E(\mathbb{R}_-)\mathcal{H} = \dim \mathcal{E}(\mathbb{R}_-)\mathbb{C}^2 =: n.$$

Let us define the mapping $J : H^2(\mathbb{R}_+) \rightarrow \mathbb{C}^2$ by the relation $Ju = (u(0), u'(0))$. Then formula (5.5) can be written as

$$(5.28) \quad h[u, u] = \int_0^\infty |u''(x)|^2 dx + \langle AJu, Ju \rangle.$$

It follows that if $u \in E(\mathbb{R}_-)\mathcal{H}$, $u \neq 0$, then $\langle AJu, Ju \rangle < 0$. Thus, the quadratic form of the operator A is negative on the subspace $JE(\mathbb{R}_-)\mathcal{H}$. This subspace has dimension N because $Ju \neq 0$ for $u \neq 0$ and hence $n \geq N$.

Conversely, pick a function $\varphi \in C^\infty(\mathbb{R}_+)$ such that $\varphi(x) = 1$ in a neighborhood of the point $x = 0$ and $\varphi(x) = 0$ for sufficiently large x and set

$$(\tilde{J}_\varepsilon a)(x) = (a_1 + a_2 x)\varphi(\varepsilon x), \quad a = (a_1, a_2), \quad \tilde{J}_\varepsilon : \mathbb{C}^2 \rightarrow H^2(\mathbb{R}_+).$$

Remark that there exists a constant $\gamma > 0$ such that $\langle Aa, a \rangle \leq -\gamma|a|^2$ for all $a \in \mathcal{E}(\mathbb{R}_-)\mathbb{C}^2$. According to (5.28) we have that

$$(5.29) \quad h[\tilde{J}_\varepsilon a, \tilde{J}_\varepsilon a] = \int_0^\infty |(\tilde{J}_\varepsilon a)''(x)|^2 dx + \langle Aa, a \rangle,$$

where

$$\int_0^\infty |(\tilde{J}_\varepsilon a)''(x)|^2 dx \leq \varepsilon|a|^2.$$

Thus, expression (5.29) is negative if $\varepsilon < \gamma$ so that $\tilde{J}_\varepsilon \mathcal{E}(\mathbb{R}_-)\mathbb{C}^2$ is a subspace of dimension n on which the quadratic form h is negative. This implies that $n \leq N$. \square

COROLLARY 5.5. *If $\alpha_0 = \alpha_1 \alpha_2 - |\alpha|^2 < 0$, then the operator H has precisely one negative eigenvalue. If $\alpha_0 > 0$, then the operator H has two negative eigenvalues for $\alpha_1 < 0$ (or equivalently $\alpha_2 < 0$) and $H \geq 0$ for $\alpha_1 > 0$ (or equivalently $\alpha_2 > 0$).*

PROOF. If $\alpha_0 = \text{Det } A < 0$, then eigenvalues μ_1 and μ_2 of the operator A have different signs. If $\alpha_0 = \text{Det } A > 0$, then $\mu_1 \mu_2 > 0$, $\alpha_1 \alpha_2 > 0$ and $\mu_1 + \mu_2 = \alpha_1 + \alpha_2$. Thus, all four numbers μ_1, μ_2 and α_1, α_2 have the same sign. \square

Let us further consider zeros of the function $\Omega(\zeta)$ on the half-axis $\zeta = k$ (or equivalently $\zeta = ik$) where $k > 0$.

PROPOSITION 5.6. *The function $\Omega(k) = 0$ for $k > 0$ if and only if $\lambda = k^4$ is an eigenvalue of the operator H .*

PROOF. Put $\zeta = k$. Recall that solution (5.24) of equation (5.25) satisfies boundary conditions (5.1) if and only if the coefficients p_1 and p_2 satisfy system (5.26). If λ is an eigenvalue of H , then $\psi(x) = \exp(-kx)$ is the eigenfunction of H and hence system (5.26) is satisfied with $p_1 = 0$ and $p_2 = 1$. It follows that $a_{12}(k) = a_{22}(k) = 0$ and hence $\Omega(k) = 0$.

Conversely, if $\Omega(k) = 0$, then system (5.26) has a nontrivial solution p_1, p_2 so that solution (5.24) of equation (5.25) satisfies boundary conditions (5.1). It remains only to show that $p_1 = 0$. Let us multiply (cf. the proof of Lemma 4.3) equation (5.25) by $\overline{u(x)}$ and integrate it over an interval $(0, r)$. Then we integrate by parts and take the imaginary part. Since the non-integral terms disappear at $x = 0$, we obtain the identity

$$\text{Im}(u'''(r)\overline{u(r)} - u''(r)\overline{u'(r)}) = 0.$$

Applying it to function (5.24) and neglecting terms exponentially decaying as $r \rightarrow \infty$, we see that $p_1 = 0$ and hence $u(x) = e^{-kx}$ is an eigenfunction of the operator H . \square

Positive eigenvalues are not exceptional for the operator H . It is easy to give simple necessary and sufficient conditions for their existence.

PROPOSITION 5.7. *A point $\lambda = k^4$, $k > 0$, is an eigenvalue of the operator H if and only if*

$$(5.30) \quad \alpha = \bar{\alpha} \quad \text{and} \quad \alpha_1 = (\alpha - k^2)k^{-1}, \quad \alpha_2 = (\alpha + k^2)k.$$

In this case

$$(5.31) \quad \Omega'(k) = -2k(\alpha - ik^2) \neq 0$$

so that k is a simple zero of the function $\Omega(\zeta)$.

PROOF. The eigenfunction of H is necessarily e^{-kx} , and boundary conditions (5.1) for this function are equivalent to equations (5.30). Differentiating (5.14) and using expressions (5.30) for α_1 and α_2 , we arrive at (5.31). \square

COROLLARY 5.8. *The operator H cannot have more than one positive eigenvalue.*

PROOF. If $\lambda = k^4$, $k > 0$, is an eigenvalue of H , then it follows from equations (5.30) that $k^4 + \alpha_0 = 0$ which determines $k > 0$ uniquely. \square

Thus, for each given $\lambda > 0$, there is a one-dimensional manifold in the four-dimensional space of parameters $\alpha, \alpha_1, \alpha_2$ parametrized by $\alpha = \bar{\alpha}$ such that the corresponding operators $H = H(\alpha, \alpha_1, \alpha_2)$ have an eigenvalue at the point λ . If k varies over \mathbb{R}_+ , then equations (5.30) determine (parametrically) a surface in the space of parameters such that the operators H have a positive eigenvalue. Note that the condition $\alpha_0 < 0$ is necessary for the existence of a positive eigenvalue.

5.3. Next we calculate the spectral measure $E(\lambda)$ of the operator H corresponding to boundary conditions (5.1).

PROPOSITION 5.9. *Suppose that $\lambda = k^4 > 0$ is not an eigenvalue of the operator H . Set*

$$(5.32) \quad s(\lambda) = \overline{\Omega(k)}/\Omega(k)$$

and

$$(5.33) \quad b(\lambda) = (\alpha_0 + 2i \operatorname{Im} \alpha k^2 + k^4)/\Omega(k)$$

where the function $\Omega(k)$ is defined by formula (5.14). Then $dE(\lambda)/d\lambda$ is the integral operator with kernel

$$(5.34) \quad dE(x, y; \lambda)/d\lambda = (2\pi)^{-1} k^{-3} \psi(x, \lambda) \overline{\psi(y, \lambda)},$$

where

$$(5.35) \quad \psi(x, \lambda) = 2^{-1}(s(\lambda)e^{ikx+\pi i/4} + e^{-ikx-\pi i/4}) - 2^{-1/2}b(\lambda)e^{-kx}.$$

PROOF. Passing to the limit $z \rightarrow \lambda \pm i0$ in equality (5.8) and using the relation between boundary values of the resolvent and the spectral measure (see formula (2.16)), we find that

$$(5.36) \quad dE(x, y; \lambda)/d\lambda = dE_{00}(x, y; \lambda)/d\lambda + (2\pi i)^{-1}(P(x, y; \lambda + i0) - \overline{P(y, x; \lambda + i0)}).$$

The kernel $P(x, y; z)$ of the operator $P(z)$ is defined by formulas (5.9) and (5.10). Since $E_{00}(\lambda) = E_{\sqrt{H_{00}}}(\sqrt{\lambda})$ where $\sqrt{H_{00}} = D^2$ with the boundary condition $u(0) = 0$, we have that

$$dE_{00}(x, y; \lambda)/d\lambda = (2\pi)^{-1}k^{-3} \sin kx \sin ky.$$

Therefore it follows from relation (5.36) that

$$(5.37) \quad \begin{aligned} dE(x, y; \lambda)/d\lambda = & -(8\pi)^{-1}k^{-3}(e^{ikx} - e^{-ikx})(e^{iky} - e^{-iky}) \\ & + (2\pi i)^{-1} \left(p_{11}(k)e^{ik(x+y)} - \overline{p_{11}(k)}e^{-ik(x+y)} \right. \\ & + p_{12}(k)e^{-k(x-iy)} - \overline{p_{21}(k)}e^{-k(x+iy)} + p_{21}(k)e^{-k(-ix+y)} - \overline{p_{12}(k)}e^{-k(ix+y)} \\ & \left. + (p_{22}(k) - \overline{p_{22}(k)})e^{-k(x+y)} \right), \end{aligned}$$

where the coefficients p_{jl} are defined by formulas (5.14) – (5.16).

Let us plug (5.35) into the right-hand side of (5.34). Taking into account definitions (5.32) and (5.33), we see that the coefficients at all terms $e^{ik(x+y)}$, $e^{-ik(x+y)}$, $e^{ik(-x+y)}$, $e^{ik(x-y)}$, $e^{-k(x-iy)}$, $e^{-k(x+iy)}$, $e^{-k(-ix+y)}$, $e^{-k(ix+y)}$ and $e^{-k(x+y)}$ are the same as in (5.37). This proves relation (5.34). \square

The functions $\psi(x, \lambda)$ defined by formula (5.35) satisfy of course boundary conditions (5.1). They are known as eigenfunctions of the continuous spectrum of the operator H . These functions describe an interaction of a plane wave $e^{-ikx - \pi i/4}$ coming from $+\infty$ with a “point” potential at $x = 0$. The coefficient $s(\lambda)$ at the reflected wave $e^{ikx + \pi i/4}$ is known as the scattering matrix. Note that if $\text{Im } \alpha = 0$ (in this case the problem is not only self-adjoint but is also real), then according to equalities (5.32) and (5.33) the functions $s(\lambda)$ and $b(\lambda)$ are related by formula (4.26). Thus, the scattering matrix $s(\lambda)$ can be recovered from the coefficient $b(\lambda)$ at the exponentially decaying mode of eigenfunction (5.35).

Proposition 5.9 implies that the positive spectrum of the operator H is absolutely continuous except, possibly, a single eigenvalue λ_0 . Moreover, we have the following result.

PROPOSITION 5.10. *The functions $s(\lambda)$ and $b(\lambda)$ are infinitely differentiable for all $\lambda > 0$. If λ_0 is a positive eigenvalue of the operator H , then*

$$(5.38) \quad s(\lambda_0) = \frac{\alpha + i\sqrt{\lambda_0}}{\alpha - i\sqrt{\lambda_0}}, \quad b(\lambda_0) = -2\frac{\sqrt{\lambda_0}}{\alpha - i\sqrt{\lambda_0}}.$$

PROOF. Let us proceed from formulas (5.32) and (5.33). Clearly, the functions $s(\lambda)$ and $b(\lambda)$ are infinitely differentiable away from the point $\lambda_0 = k_0^4$ (the eigenvalue of H) where $\Omega(k_0) = 0$. As far as their behavior at λ_0 is concerned, the result about $s(\lambda)$ follows from formulas (5.31) and

$$s(\lambda_0) = \overline{\Omega'(k_0)}/\Omega'(k_0).$$

To consider $b(\lambda)$, we take additionally into account that according to Proposition 5.7 a point $\lambda_0 = k_0^4$ is a positive eigenvalue of the operator H if and only if conditions

(5.30) are satisfied for $k = k_0$. Therefore the numerator in (5.33) equals zero at $k = k_0$ and its derivative equals $4k_0^3$. This yields relation (5.38) for $b(\lambda_0)$. \square

We emphasize that both functions $is(\lambda_0)e^{ik_0x} + e^{-ik_0x}$ and e^{-k_0x} satisfy boundary condition (5.1) if λ_0 is an eigenvalue.

REMARK 5.11. The function $b(\lambda)$ might have zero only at one point $\lambda_0 = -\alpha_0$. According to Proposition 5.7 this condition is satisfied if λ_0 is an eigenvalue of H . However according to Proposition 5.10 in this case $b(\lambda_0) \neq 0$. On the contrary, $b(\lambda_0) = 0$ if $\lambda_0 = -\alpha_0$, $\alpha = \bar{\alpha}$, but the last two conditions (5.30) are violated at $k = k_0$. For example, if $\alpha = k_0^2$, $\alpha_1 = 0$ but $\alpha_2 \neq 2k_0$, then $b(\lambda_0) = 0$ for $\lambda_0 = k_0^4$.

5.4. Now we are in a position to establish an expansion in eigenfunctions of the operator H .

THEOREM 5.12. Let $\psi_-(x, \lambda) = \psi(x, \lambda)$ be defined by formulas (5.32), (5.33), (5.35), and let

$$(5.39) \quad \psi_+(x, \lambda) = \overline{s(\lambda)}\psi_-(x, \lambda).$$

Define on the set $L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$ the mappings \mathcal{F}_\pm by the formula

$$(5.40) \quad (\mathcal{F}_\pm f)(\lambda) = 2^{-1}(2\pi)^{-1/2}\lambda^{-3/8} \int_0^\infty \overline{\psi_\pm(x, \lambda)} f(x) dx.$$

These mappings extend to bounded operators on the space $L_2(\mathbb{R}_+)$ and satisfy relations (2.12) and (2.13).

PROOF. Intertwining property (2.13) holds because the functions $\psi_\pm(x, \lambda)$ satisfy the equation $\psi_\pm^{(4)} = \lambda\psi_\pm$ and boundary conditions (5.1). The second relation (2.12) is obtained by integration of representation (5.34) over $\lambda \in \mathbb{R}_+$. Now for the proof of the first relation (2.12), we have to check that the kernel of the operator \mathcal{F}_\pm^* is trivial. Supposing that $\mathcal{F}_\pm^*g = 0$ and hence $\mathcal{F}_\pm^*E_A(\alpha, \beta)g = 0$ for all $\alpha, \beta \in \mathbb{R}_+$, we have

$$\int_\alpha^\beta \psi(x, \lambda)g(\lambda)d\lambda = 0, \quad \forall x \geq 0.$$

Let us differentiate twice this relation, use (5.35) and set $x = 0$. Since α and β are arbitrary, we obtain that

$$\begin{aligned} (s(\lambda) - i - 2^{1/2}e^{-\pi i/4}b(\lambda))g(\lambda) &= 0, & (s(\lambda) + i - 2^{1/2}e^{\pi i/4}b(\lambda))g(\lambda) &= 0, \\ (s(\lambda) - i + 2^{1/2}e^{-\pi i/4}b(\lambda))g(\lambda) &= 0 \end{aligned}$$

for a.e. $\lambda > 0$. If $g(\lambda) \neq 0$ for some λ , then the first and third equations imply that $s(\lambda) = i$ and $b(\lambda) = 0$ which contradicts the second equation. Thus, $g(\lambda) = 0$ for a.e. $\lambda > 0$. \square

REMARK 5.13. If $\alpha = \bar{\alpha}$, then $\psi_+(x, \lambda) = \overline{\psi_-(x, \lambda)}$.

Consider now the operator $H_0 = D^4$ corresponding to boundary conditions (4.22). Its eigenfunctions are defined by formula (4.23). It follows from formulas (5.19) and (5.20) that the spectral measure $E_0(\lambda)$ of the operator H_0 satisfies relation (5.34) where the role of ψ is played by ψ_0 . Theorem 5.12 applies of course to the operator \mathcal{F}_0 defined by the formula

$$(5.41) \quad (\mathcal{F}_0 f)(\lambda) = 2^{-1}(2\pi)^{-1/2}\lambda^{-3/8} \int_0^\infty \psi_0(x, \lambda)f(x)dx.$$

Moreover, the operator \mathcal{F}_0 is unitary because the operator H_0 does not have eigenvalues.

Theorem 5.12 allows us to construct directly the time-dependent scattering theory in the same way as for the second order differential operators. We need also the following auxiliary assertion.

LEMMA 5.14. *If $u \in C_0^\infty(\mathbb{R}_+)$, then*

$$(5.42) \quad \lim_{t \rightarrow \pm\infty} \int_0^\infty dx \left| \int_0^\infty \exp(\mp ikx - ik^4 t) u(k) dk \right|^2 = 0$$

and

$$(5.43) \quad \lim_{|t| \rightarrow \infty} \int_0^\infty dx \left| \int_0^\infty \exp(-kx - ik^4 t) u(k) dk \right|^2 = 0.$$

PROOF. Both relations (5.42) and (5.43) are obtained by a direct integration by parts which shows that the integral over k is bounded by $C(x + |t|)^{-1}$. \square

THEOREM 5.15. *The wave operators $W_\pm = W_\pm(H, H_0)$ exist, are complete and satisfy the equality $W_\pm = \mathcal{F}_\pm^* \mathcal{F}_0$. The scattering operator \mathcal{S} for the pair H_0, H acts in the space $L_2(\mathbb{R}_+)$ as multiplication by the function $s(\lambda)$ defined by formulas (5.14), (5.32).*

PROOF. According to Theorem 5.12 all results about the wave operators W_\pm follow from the relation

$$(5.44) \quad \lim_{t \rightarrow \pm\infty} \|(\mathcal{F}_\pm^* - \mathcal{F}_0^*) \exp(-iAt)g\| = 0$$

where $g(k)$ is an arbitrary function from $L_2(\mathbb{R}_+)$. It suffices to check (5.44) on the set $C_0^\infty(\mathbb{R}_+)$. Using (5.40), (5.41), we obtain that

$$(5.45) \quad \begin{aligned} & ((\mathcal{F}_\pm^* - \mathcal{F}_0^*) \exp(-iAt)g)(x) \\ &= 2^{-1}(2\pi)^{-1/2} \int_0^\infty (\psi_\pm(x, \lambda) - \psi_0(x, \lambda)) \exp(-i\lambda t) g(\lambda) \lambda^{-3/8} d\lambda. \end{aligned}$$

Let us check (5.44), for example, for the sign “−”. It follows from (4.23) and (5.35) that

$$\psi_-(x, \lambda) - \psi_0(x, \lambda) = 2^{-1}(s(\lambda) - 1)e^{ikx + \pi i/4} - 2^{-1/2}(b(\lambda) - 1)\exp(-kx).$$

The contributions to (5.45) of the first and second terms in the right-hand side tend in $L_2(\mathbb{R}_+)$ to zero as $t \rightarrow -\infty$ according to relations (5.42) and (5.43), respectively.

Equality $W_\pm = \mathcal{F}_\pm^* \mathcal{F}_0$ implies that the scattering operator \mathcal{S} satisfies relation (2.22). It remains to remark that $(\mathcal{F}_+ f)(\lambda) = s(\lambda)(\mathcal{F}_- f)(\lambda)$ according to (5.39). \square

REMARK 5.16. Since the operator \mathcal{F}_0 as well as the Fourier transform are bounded operators, it follows from formula (4.23) that the integral operator T with kernel $\exp(-kx)$ is bounded in $L_2(\mathbb{R}_+)$. This result is a by-product of our considerations, but it is not of course new. Indeed, we have that

$$\|Tu\|^2 = (\mathcal{C}u, u)$$

where \mathcal{C} is the integral operator with kernel $(k + k')^{-1}$ known as Carleman’s operator. A proof of its boundedness can be found, e.g., in [12]. We note also that boundedness of the Fourier transform and of the operator T imply that relations (5.42) and (5.43) remain true for all $u \in L_2(\mathbb{R}_+)$.

6. The perturbation determinant and the spectral shift function

6.1. Mathematical theory of the spectral shift function was constructed by M. Kreĭn. Let us recall here briefly its basic notions (see [7] and [16], for details). Suppose that self-adjoint operators H_0 and H are semibounded from below and that the difference of their resolvents belongs to the trace class.

The (generalized) perturbation determinant $D(z)$ for the pair H_0, H is defined by the equation

$$(6.1) \quad \text{Tr}(R(z) - R_0(z)) = -D'(z)D(z)^{-1}, \quad z \notin \sigma(H_0) \cup \sigma(H),$$

which fixes $D(z)$ up to a constant factor. We set

$$(6.2) \quad \ln D(z) = \int_{z_0}^z \text{Tr}(R_0(z') - R(z'))dz'$$

where z_0 is some real point lying below

$$\nu = \min\{\inf \sigma(H_0), \inf \sigma(H)\}$$

and the integral is taken over some contour in the upper (lower) half-plane if $\text{Im } z > 0$ ($\text{Im } z < 0$). Then the function $\ln D(z)$ is determined up to a real constant (and hence the function $D(z)$ is determined up to a constant positive factor) which is inessential. Clearly, $\arg D(z) = 0$ for $z = \bar{z} < \nu$ and $D(\bar{z}) = \overline{D(z)}$.

The limit of $\arg D(\lambda + i\varepsilon)$ exists for a.e. $\lambda \in \mathbb{R}$, and the spectral shift function $\xi(\lambda)$ for the pair H_0, H is defined by the formula

$$(6.3) \quad \xi(\lambda) = \pi^{-1} \arg D(\lambda + i0).$$

The function $\xi(\lambda)$ assumes constant integral values on component intervals of the set of common regular points of the operators H_0 and H , $\xi(\lambda) = 0$ if $\lambda < \nu$ and

$$(6.4) \quad \xi(\lambda + 0) - \xi(\lambda - 0) = n_0 - n$$

if λ is an isolated eigenvalue of multiplicity n_0 of the operator H_0 and of multiplicity n of the operator H . The spectral shift function satisfies the condition

$$\int_{-\infty}^{\infty} |\xi(\lambda)|(1 + \lambda^2)^{-1} d\lambda < \infty.$$

The trace formula

$$\text{Tr}(\varphi(H) - \varphi(H_0)) = \int_{-\infty}^{\infty} \xi(\lambda)\varphi'(\lambda)d\lambda$$

holds for functions $\varphi \in C^2(\mathbb{R})$ such that

$$(\lambda^2 \varphi'(\lambda))' = O(\lambda^{-1-\varepsilon}), \quad \lambda \rightarrow +\infty,$$

for some $\varepsilon > 0$. In particular, the trace formula is true for the function $\varphi(\lambda) = (\lambda - z)^{-1}$ when $\varphi(H) = R(z)$.

For the pair H_0, H , wave operators (2.14) exist, are complete and the relation between the scattering matrix $S(\lambda)$ and the spectral shift function is given by the Birman-Kreĭn formula [5]

$$(6.5) \quad \text{Det } S(\lambda) = e^{-2\pi i \xi(\lambda)}.$$

6.2. Now we are in a position to construct explicitly the perturbation determinant and the spectral shift function for the pair H_0, H considered in the previous section. According to equalities (5.8) and (5.19) we have

$$(6.6) \quad \text{Tr}(R(z) - R_0(z)) = \text{Tr} P(z) - \text{Tr} P_0(z).$$

The kernel $P(x, y; z)$ of the operator $P(z)$ is determined by formulas (5.9) and (5.10) so that

$$P(x, x; z) = p_{11}(\zeta)e^{2i\zeta x} + (p_{12}(\zeta) + p_{21}(\zeta))e^{(-1+i)\zeta x} + p_{22}(\zeta)e^{-2\zeta x}.$$

It follows that

$$\text{Tr} P(z) = \int_0^\infty P(x, x; z)dx = (2\zeta)^{-1}(ip_{11}(\zeta) + (1+i)(p_{12}(\zeta) + p_{21}(\zeta)) + p_{22}(\zeta)).$$

Using formulas (5.15) and (5.16) for the coefficients p_{ji} , we find that

$$\text{Tr} P(z) = (4\zeta^4\Omega(\zeta))^{-1}(\alpha_0 + 2i \text{Re} \alpha \zeta^2 + 2(1+i)\alpha_1\zeta^3 + 3\zeta^4).$$

Similarly, it follows from (5.20) that

$$(6.7) \quad \text{Tr} P_0(z) = (4z)^{-1}.$$

Therefore equality (6.6) yields the following result.

PROPOSITION 6.1. *Suppose that boundary conditions (5.1) are satisfied. Then*

$$(6.8) \quad \text{Tr}(R(z) - R_0(z)) = -4^{-1}\zeta^{-3}\Omega(\zeta)^{-1}\Omega'(\zeta)$$

where the function Ω is defined by formula (5.14).

Let us now consider boundary conditions (5.2) and (5.3). Remark that formally conditions (5.2) can be obtained from (5.1) if we set $\alpha_1 = N$, replace α by $-\alpha N$, replace α_2 by $|\alpha|^2 N + \alpha_2$ and pass to the limit $N \rightarrow \infty$. We plug these expressions into (5.14) and observe that the limit of the right-hand side of (6.8) is determined only by the terms containing the factor N . Thus, the equation (6.8) is true if we set

$$(6.9) \quad \Omega(\zeta) = \alpha_2 + (1-i)|\alpha|^2\zeta - 2i \text{Re} \alpha \zeta^2 - (1+i)\zeta^3.$$

Similarly, in case (5.3) we set $\alpha = 0$, $\alpha_2 = N$ and take the limit $N \rightarrow \infty$. This yields

$$(6.10) \quad \Omega(\zeta) = \alpha_1 + (1-i)\zeta.$$

All results obtained in Section 5 for boundary conditions (5.1) can easily be carried over to cases (5.2) and (5.3). In particular, formula (5.32) remains true.

Comparing equations (6.1) and (6.8), we obtain

PROPOSITION 6.2. *The perturbation determinant for the pair H_0, H is given by the equality*

$$(6.11) \quad D(z) = \Omega(\zeta), \quad z = \zeta^4.$$

Since all functions (5.14), (6.9) and (6.10) satisfy the condition $\Omega(e^{\pi i/4}k) > 0$ for large $k > 0$, we can set $\arg \Omega(e^{\pi i/4}k) = 0$ for such k . Thus, $\Omega(\zeta)$ determines by formulas (6.3), (6.11) the spectral shift function $\xi(\lambda)$ for the pair H_0, H . Clearly, $\xi(\lambda) = 0$ below the lowest eigenvalue of H . Proposition 5.3 implies that the spectral shift function has a jump -1 at a simple eigenvalue of H and a jump -2 at an eigenvalue of multiplicity 2. This is of course consistent with general formula (6.4). In view of definition (6.3), relation (5.32) reduces to formula (6.5).

Let us write formulas (5.14), (6.9) and (6.10) in a unified way as

$$(6.12) \quad D(z) = \sum_{j=0}^4 \omega_j \zeta^j, \quad z = \zeta^4.$$

We put

$$(6.13) \quad \gamma_0 = \min_j \{j : \omega_j \neq 0\}, \quad \gamma_1 = \max_j \{j : \omega_j \neq 0\}.$$

Results about the behavior of $\xi(\lambda)$ for $\lambda > 0$ are collected in the following assertion.

PROPOSITION 6.3. *For $\lambda > 0$, the spectral shift function for the pair H_0, H is infinitely differentiable away from a positive eigenvalue λ_0 of the operator H (if it exists). The limits of $\xi(\lambda)$ as $\lambda \rightarrow \lambda_0 \pm 0$ exist and*

$$(6.14) \quad \xi(\lambda_0 + 0) - \xi(\lambda_0 - 0) = -1.$$

If an interval $(\lambda_1, \lambda_2) \subset \mathbb{R}_+$ does not contain an eigenvalue of H , then

$$(6.15) \quad \xi(\lambda_2) - \xi(\lambda_1) = \pi^{-1} \int_{k_1}^{k_2} \operatorname{Im}(\Omega(k)^{-1} \Omega'(k)) dk, \quad \lambda_j = k_j^4.$$

Moreover, there exists the limit $\xi(+0)$,

$$(6.16) \quad \delta := \xi(+0) - \xi(-0) = -\gamma_0/4$$

and

$$(6.17) \quad \lim_{\lambda \rightarrow +\infty} \xi(\lambda) = -\gamma_1/4.$$

PROOF. According to Proposition 5.6 the perturbation determinant $D(\lambda + i0) = 0$ if and only if $\lambda = \lambda_0$ is an eigenvalue of the operator H . Since $D(\lambda + i0)$ is a C^∞ -function for $\lambda > 0$, its argument is also a C^∞ -function away from the point λ_0 . According to Proposition 5.7, λ_0 is a simple zero of the function $D(z)$ so that

$$D(z) = d(z - \lambda_0) + O(|z - \lambda_0|^2), \quad z \rightarrow \lambda_0, \quad \operatorname{Im} z \geq 0,$$

for some $d \neq 0$. Therefore the function $\arg D(\lambda + i0)$ has finite limits as $\lambda \rightarrow \lambda_0 \pm 0$ and the variation of $\arg D(z)$ as z passes from $\lambda_0 - \varepsilon$ to $\lambda_0 + \varepsilon$ in the clockwise direction over a semi-circle $C_\varepsilon^+(\lambda_0) = \{|z - \lambda_0| = \varepsilon, \operatorname{Im} z \geq 0\}$ equals

$$(6.18) \quad \operatorname{var}_{C_\varepsilon^+(\lambda_0)} \arg D(z) = -\pi + o(1)$$

as $\varepsilon \rightarrow 0$. This proves formula (6.14). Relation (6.15) is a direct consequence of definition (6.3) and relation (6.11).

The existence of $\xi(+0)$ and relation (6.16) follow from formula (6.12) and the equality

$$(6.19) \quad \operatorname{var}_{C_\varepsilon^+(0)} \arg D(z) = -\pi\gamma_0/4 + o(1)$$

as $\varepsilon \rightarrow 0$. Similarly, relation (6.17) follows from formula (6.12) and the equality

$$(6.20) \quad \operatorname{var}_{C_R^+(0)} \arg D(z) = -\pi\gamma_1/4 + o(1)$$

as $R \rightarrow \infty$. □

If boundary conditions (5.1) are satisfied, then $\gamma_1 = 4$, but γ_0 might equal (see (5.14)) any number between 0 and 4. In particular, if $\alpha = \alpha_1 = \alpha_2 = 0$, then $D(z) = -z$ and $\xi(\lambda) = 0$ for $\lambda < 0$ and $\xi(\lambda) = -1$ for $\lambda > 0$.

7. Zero-energy resonances

Here we analyse resolvent singularities and the behavior of the spectral shift function at the bottom of the continuous spectrum. The notion of a zero-energy resonance is introduced and discussed.

7.1. Resolvents of all operators considered in Section 5 admit asymptotic expansions (for fixed x and y) in powers of ζ as $|\zeta| \rightarrow 0$. Let us find their singular parts. Below the symbol “ \simeq ” means an equality valid up to regular terms.

According to formula (5.7)

$$(7.1) \quad R_{00}(x, y; z) = 2^{-1/2}xy(-z)^{-1/4} + 12^{-1}(|x - y|^3 - (x + y)^3) + O(|z|^{1/4}),$$

according to formula (5.20)

$$(7.2) \quad P_0(x, y; z) = -2^{-1/2}xy(-z)^{-1/4} + 2^{-1}xy(x + y) + O(|z|^{1/4})$$

and according to formula (5.21)

$$(7.3) \quad P_1(x, y; z) \simeq 2^{-1/2}(-z)^{-3/4} - 2^{-3/2}(x + y)^2(-z)^{-1/4}.$$

It is slightly more difficult to find the singular part of kernel $P(x, y; z)$. It follows from equations (5.9), (5.10) and (5.15) that

$$(7.4) \quad P(x, y; z) = 2^{-3/2}e^{-\pi i/4}\zeta^{-3}\Omega(\zeta)^{-1} \sum_{n=0}^{\infty} (n!)^{-1} L_n(x, y; \zeta) \zeta^n$$

where

$$\begin{aligned} L_n(x, y; \zeta) &= \mathbf{p}_{11}(\zeta)i^n(x + y)^n + \mathbf{p}_{12}(\zeta)(-x + iy)^n \\ &+ \mathbf{p}_{21}(\zeta)(ix - y)^n + \mathbf{p}_{22}(\zeta)(-1)^n(x + y)^n \end{aligned}$$

and the coefficients $\mathbf{p}_{jl}(\zeta)$, $j, l = 1, 2$, are defined by formulas (5.16). In particular, we have

$$(7.5) \quad L_0(x, y; \zeta) = 2(1 + i)\alpha_1\zeta^3 + 2\zeta^4,$$

$$(7.6) \quad L_1(x, y; \zeta) = -2(1 + i)(\alpha x + \bar{\alpha}y)\zeta^2 + 2(-1 + i)(x + y)\zeta^4$$

and

$$(7.7) \quad L_2(x, y; \zeta) = -4i\alpha_0xy + 2(\alpha x^2 + 4\operatorname{Re} \alpha xy + \bar{\alpha}y^2)\zeta^2 + 2(i - 1)\alpha_1(x^2 + y^2)\zeta^3 - 4ixy\zeta^4.$$

Let first $\alpha_0 \neq 0$. Then $\Omega(0) = \alpha_0 \neq 0$ and the only singular term in (7.4) comes from the first term in the right-hand side of (7.7) which yields

$$(7.8) \quad P(x, y; z) \simeq -2^{-1/2}xy(-z)^{-1/4}, \quad \alpha_0 \neq 0.$$

If $\alpha_0 = 0$ but $\alpha_2 \neq 0$, then $\mathbf{p}_{jl}(\zeta) = O(|\zeta|^2)$, $\Omega(\zeta) = (1 - i)\alpha_2\zeta(1 + O(|\zeta|))$ as $|\zeta| \rightarrow 0$ and singular terms in (7.4) come from the first terms in the right-hand sides of (7.5) and (7.6). Thus, we have

$$(7.9) \quad P(x, y; z) \simeq 2^{-3/2}\alpha_2^{-1}(2\alpha_1 - \alpha x - \bar{\alpha}y)(-z)^{-1/4}, \quad \alpha_0 = 0, \quad \alpha_2 \neq 0.$$

If $\alpha_0 = \alpha_2 = 0$ (and hence $\alpha = 0$) but $\alpha_1 \neq 0$, then $\mathbf{p}_{jl}(\zeta) = O(|\zeta|^2)$, $\Omega(\zeta) = -(1 + i)\alpha_1\zeta^3(1 + O(|\zeta|))$ as $|\zeta| \rightarrow 0$ and singular terms in (7.4) come from (7.5) and the second term in the right-hand side of (7.6). Thus, in this case we have

$$(7.10) \quad P(x, y; z) \simeq 2^{-1/2}(-z)^{-3/4} - (\alpha_1^{-1}2^{-1/2}(x + y) + 2^{-3/2}(x^2 + y^2))(-z)^{-1/4}.$$

Let finally $\alpha = \alpha_1 = \alpha_2 = 0$. Then it follows from formula (5.17) that

$$(7.11) \quad P(x, y; z) \simeq 2^{-1/2}(-z)^{-3/4} - 2^{-1}(x+y)(-z)^{-1/2} + 2^{-1/2}xy(-z)^{-1/4}.$$

7.2. Here we discuss different types of zero-energy resonances. Let us distinguish several cases.

¹⁰ The operator H_0 does not have zero-energy resonances. Since singular terms in (7.1) and (7.2) are compensated in (5.19), the resolvent kernel $R_0(x, y; z)$ is a continuous function as $z \rightarrow 0$ and

$$R_0(x, y; 0) = 12^{-1}(|x - y|^3 - (x + y)^3) + 2^{-1}xy(x + y).$$

Linear functions (except zero) do not satisfy boundary conditions (4.22). All boundary conditions (5.1), (5.2) and (5.3) change the domain of quadratic form of the operator H_0 , and hence the corresponding operators H cannot be considered as small perturbations of H_0 .

²⁰ The operator H_{00} has a zero-energy resonance. Formula (7.1) shows that its resolvent kernel has the singularity $2^{-1/2}xy(-z)^{-1/4}$ as $z \rightarrow 0$. The function $u(x) = x$ satisfies boundary conditions (4.22). According to equality (6.10) where $\alpha_1 = 0$ the perturbation determinant for the pair H_0, H_{00} equals $D_{00}(z) = (-z)^{1/4}$, and the spectral shift function equals 0 for $\lambda < 0$ and $-1/4$ for $\lambda > 0$. In view of formula (7.1) it is natural to say that the operator H_{00} has a quarter-bound state at zero energy. The operator H_{00} belongs to family (5.3) for $\alpha_1 = 0$. One negative eigenvalue appears for an arbitrary $\alpha_1 < 0$.

Next we consider the operator H with boundary conditions (5.1). According to formula (5.8) the singular part of its resolvent equals the sum of singular parts of (7.1) and of the kernel $P(x, y; z)$. The jump δ of the spectral shift function $\xi(\lambda)$ at the point $\lambda = 0$ is determined by formula (6.16).

³⁰ If $\alpha_0 \neq 0$, then the operator H does not have zero-energy resonances. Since the singularities in (7.1) and (7.8) are compensated, the kernel $R(x, y; z)$ is a continuous function as $z \rightarrow 0$. Linear functions $u(x) = Ax + B$ satisfy boundary conditions (5.1) only for $A = B = 0$. Since $\Omega(0) = \alpha_0 \neq 0$, the spectral shift function is continuous at the point $\lambda = 0$. According to Proposition 5.4 new negative eigenvalues of the operator H cannot appear under small perturbations of the coefficients α, α_1 and α_2 .

⁴⁰ Let $\alpha_0 = 0$ but $\alpha_2 \neq 0$. Then formulas (7.1), (7.9) show that the resolvent kernel has a singularity at $z = 0$ of the same order $-1/4$ as $R_{00}(z)$. The linear function $u(x) = \alpha_2 x - \bar{\alpha}$ satisfies boundary conditions (5.1). Now $\omega_0 = 0$ but $\omega_1 = (1 - i)\alpha_2 \neq 0$ in (6.12) and hence $\gamma_0 = 1$. It follows from (6.16) that $\delta = -1/4$. Thus, the operator H has a zero-energy resonance of the same “strength” (that is $1/4$ -bound state) as the operator H_{00} .

⁵⁰ Let $\alpha = \alpha_2 = 0$ but $\alpha_1 \neq 0$. Then formula (7.10) shows that its resolvent kernel has the singularity $2^{-1/2}(-z)^{-3/4}$ as $|z| \rightarrow 0$. The equation $u^{(4)}(x) = 0$ has a solution $u(x) = 1$ satisfying boundary condition (5.1). Now $\omega_0 = \omega_1 = \omega_2 = 0$ but $\omega_3 = -(1 + i)\alpha_1 \neq 0$ and hence $\gamma_0 = 3$. It follows from (6.16) that $\delta = -3/4$. To comply with these results, we say that the operator H has a $3/4$ -bound state at energy zero.

In the cases ⁴⁰ and ⁵⁰, we have that $\text{Det } A = 0$ but $A \neq 0$ so that the matrix A has exactly one zero eigenvalue. Therefore the operator H has an additional negative eigenvalue for an arbitrary negative perturbation of A .

⁶⁰ Let $\alpha = \alpha_1 = \alpha_2 = 0$. Then both functions $u(x) = 1$ and $u(x) = x$ satisfy boundary conditions (5.1). It follows from formula $\Omega(\zeta) = -\zeta^4$ that $\delta = -1$. The operator H has both $3/4$ - and $1/4$ -bound states at energy zero. This is consistent with formula $\delta = -1$ as well as with the fact that the operator H has two negative eigenvalues for an arbitrary matrix $A < 0$. According to (7.11) the resolvent kernel contains more singularities than in the previous cases.

Families (5.2) and (5.3) can be considered in a similar but simpler way. We discuss only the operator H_1 .

⁷⁰ Let boundary conditions (5.18) be satisfied. Formula (7.3) shows that the singularity $2^{-1/2}(-z)^{-3/4}$ of the resolvent kernel is the same as in case 5⁰. The function $u(x) = 1$ satisfies conditions (5.18). According to equality (6.9) where $\alpha = \alpha_2 = 0$ the perturbation determinant for the pair H_0, H_1 equals $D_1(z) = (-z)^{3/4}$, and the spectral shift function equals 0 for $\lambda < 0$ and $-3/4$ for $\lambda > 0$. Thus, the operator H has a $3/4$ -bound state at energy zero. As follows from (5.6), the operator H corresponding to boundary conditions (5.2) where $\alpha = 0$ has a negative eigenvalue for all $\alpha_2 < 0$.

7.3. We finish with an analogue of the Levinson theorem. Consider the closed contour which consists of a small circle $C_\varepsilon = \{|z| = \varepsilon\}$, a big circle $C_R = \{|z| = R\}$ and two intervals (ε, R) lying on the upper and lower edges of the cut along $[0, \infty)$. Moreover, if the operator H has a positive eigenvalue λ_0 we go over it by small semi-circles $C_\varepsilon^\pm(\lambda_0)$ where $|z - \lambda_0| = \varepsilon$ and $\pm \operatorname{Im} z \geq 0$ (lying in the upper and lower half-planes). Let us pass this contour in the positive direction and apply the argument principle to the function $D(z)$. Taking into account the direction of motion and the identity $\overline{D(z)} = D(\bar{z})$, we obtain that

$$(7.12) \quad \operatorname{var}_{C_\varepsilon^+} \arg D(z) - \operatorname{var}_{C_R^+} \arg D(z) + \arg D(R + i0) - \arg D(\varepsilon + i0) = \pi N_-$$

where N_- is the number of negative eigenvalues of the operator H . Moreover, if H has a positive eigenvalue, than the term

$$(7.13) \quad \arg D(\lambda_0 - \varepsilon + i0) - \arg D(\lambda_0 + \varepsilon + i0) + \operatorname{var}_{C_\varepsilon^+(\lambda_0)} \arg D(z)$$

should be added to the left-hand side.

Let us choose $\arg s(\lambda)$ as a continuous function of $\lambda > 0$. According to formulas (5.32), (6.3) we have that

$$\begin{aligned} \arg D(R + i0) - \arg D(\lambda_0 + \varepsilon + i0) &= -2^{-1}(\arg s(R) - \arg s(\lambda_0 + \varepsilon)), \\ \arg D(\lambda_0 - \varepsilon + i0) - \arg D(\varepsilon + i0) &= -2^{-1}(\arg s(\lambda_0 - \varepsilon) - \arg s(\varepsilon)). \end{aligned}$$

Taking the limits $\varepsilon \rightarrow 0$, $R \rightarrow \infty$ and using relations (6.18) – (6.20), we finally obtain that

$$(7.14) \quad \arg s(\infty) - \arg s(+0) = -2\pi N + \pi(\gamma_1 - \gamma_0)/2$$

where N is the total number of eigenvalues (including eventually a positive eigenvalue) of the operator H and the numbers γ_0, γ_1 are defined by formula (6.13). For example, for boundary conditions (5.1), $\gamma_1 = 4$ and the number γ_0 has been computed in the previous subsection. We emphasize that an isolated eigenvalue and an eigenvalue embedded in the continuous spectrum give the same contributions to the Levinson formula (7.14).

References

- [1] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa **2** no. 4 (1975), 151-218.
- [2] R. Beals, P. Deift and C. Tomei, *Direct and inverse scattering on the line*, Math. surveys and monographs, N 28, Amer. Math. Soc., Providence, R. I., 1988.
- [3] M. Sh. Birman, *On the spectrum of singular boundary-value problems*, Matem. sb. **55**, no. 2 (1961), 125-174 (Russian); English transl., Eleven Papers on Analysis, Amer. Math. Soc. Transl. (2), vol. 53, Amer. Math. Soc., Providence, RI, 1966, 23-60.
- [4] M. Sh. Birman, *Scattering problems for differential operators with constant coefficients*, Funct. Anal. Appl. **3** no. 3 (1969), 167-180.
- [5] M. Sh. Birman and M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Soviet Math. Dokl. **3** (1962), 740-744.
- [6] M. Sh. Birman and D. R. Yafaev, *A general scheme in the stationary theory of scattering*, Problemy Math. Phys. **12** (1987), 89-117. English transl., Amer. Math. Soc. Transl. (Ser.2) **157** (1993), 87-112.
- [7] M. Sh. Birman and D. R. Yafaev, *The spectral shift function. The papers of M. G. Kreĭn and their further development*, St. Petesburg Math. J. **4** no. 5 (1993), 833-870.
- [8] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.
- [9] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Inc., Englewood Cliffs, N. Y., 1962.
- [10] S. T. Kuroda, *Scattering theory for differential operators*, J. Math. Soc. Japan **25** no. 1,2 (1973), 75-104, 222-234.
- [11] S. T. Kuroda, *An introduction to scattering theory*, Lect. Notes Series No. 51, Aarhus University, 1978.
- [12] S. G. Power, *Hankel operators on Hilbert space*, Research Notes in Math., Vol 64, Pitman, Boston, 1982.
- [13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1978.
- [14] L. D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Amer. Math. Soc. Transl. (Ser.2) **65** (1967), 139-166.
- [15] D. R. Yafaev, *The virtual level of the Schrödinger equation*, Zap. Naučhn. Sem. LOMI **51** (1975), 203-216. English transl., J. Sov. Math. **11** (1979), 501-510.
- [16] D. R. Yafaev, *Mathematical scattering theory*, Amer. Math. Soc., Providence, Rhode Island, 1992.
- [17] D. R. Yafaev, *On a zero-range interaction of a quantum particle with the vacuum*, J. Phys. A, **25** (1992), 963-978.
- [18] D. R. Yafaev, *Scattering theory: some old and new problems*, Springer Lecture Notes Math. **1735**, 2000.

IRMAR, UNIVERSITÉ DE RENNES I, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE
 E-mail address: yafaev@univ-rennes1.fr